Topological Quantum Computation by Fractional Quantum Hall Effect

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Topological Order and Anyons
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Georgiev L S, Phys Rev B, 2006, 74(23): 235112.
Nayak C et al, Rev Mod Phys, 2008, 80(3): 1083.

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Basic Ideas of Topological Quantum Computation

Topological properties are global properties and thus are resilient against local perturbations.

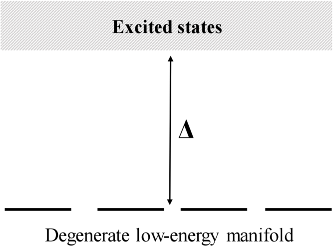

Figure: Lahtinen V et al, SciPost Physics, 2017, 3(3).

Ground state is degenerate and separated by topological properties.
Degenerate ground states can not jump without going to excited states.

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Integer and Fractional Quantum Hall Effect

Wave function in integer quantum Hall effect under symmetric gauge:

$$\psi_m \sim z^m e^{-|z|^2/4l_B^2}$$

Considering interaction between electrons yields fractional quantum Hall effect. Two-body wave function becomes

$$\psi \sim (z_1 + z_2)^M (z_1 - z_2)^m e^{-\left(|z_1|^2+|z_2|^2\right)/4l_B^2}$$

Many-particle wave function (Laughlin state):

$$\psi(z_i) = \prod_{i<j} (z_i - z_j)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

Moore-Read State and Quasiholes

Moore-Read state (or Pfaffian state) is defined by

$$\tilde{\psi}_{MR}(z) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m$$

Pfaffian is the "square root" of a determinant, e.g. 4 particles:

$$\text{Pf}\left(\frac{1}{z_i - z_j}\right) = \frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} + \frac{1}{z_1 - z_3}\frac{1}{z_4 - z_2} + \frac{1}{z_1 - z_4}\frac{1}{z_2 - z_3}$$

Quasiholes are excitations of Moore-Read state

$$\tilde{\psi}(z) = \prod_k (z_k - \eta) \, \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m$$

Non-abelian Anyons and Plasma Analogy

Quasiholes are non-abelian anyons. This can be seen from the plasma analogy:

$$\mathcal{Z} = \int \exp \sum_{i,j} \left[\log\left(|z_i - z_j|^2 \right) - \frac{1}{2l_B^2} |z_i|^2 + m \log\left(|z_i - \eta_j|^2 \right) \right]$$

and computing the Berry connection

$$\mathcal{A}_\eta(\eta, \overline{\eta}) = -\frac{i}{2\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \eta} = \frac{i}{2} \frac{\partial \log(\mathcal{Z})}{\partial \eta}$$

$$\mathcal{A}_{\overline{\eta}}(\eta, \overline{\eta}) = \frac{i}{2} \frac{\partial \log(\mathcal{Z})}{\partial \overline{\eta}}$$

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View from Composite Fermions

- Our focus is $\nu = 5/2$ fractional quantum Hall state, which consists of a fully filled lowest Landau level and half filled second Landau level.
- The Moore-Read state arises when these composite fermions pair up and condense, forming a p-wave superconductor.
- In common with all superconductors, there are vortices, which will be our quasiholes.
- The vortices have zero modes. They can be thought of as fermion-vortex bound states. In p-wave superconductors, this zero mode is Majorana.

Hilbert Space Generated by Majorana Zero Modes

Majorana zero modes satisfy Clifford algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

Vortices pair up and become Dirac fermion:

$$\Psi_k = \frac{1}{2}(\gamma_{2k-1} + i\gamma_{2k}) \quad k = 1, \ldots, n$$

These Dirac fermions satisfy

$$\left\{\Psi_k, \Psi_l^\dagger\right\} = \delta_{kl} \quad \text{and} \quad \{\Psi_k, \Psi_l\} = \left\{\Psi_k^\dagger, \Psi_l^\dagger\right\} = 0$$

Hilbert Space Generated by Majorana Zero Modes

Many-particle Hilbert space can be constructed as

$$|\Omega\rangle$$

$$\Psi_k^\dagger|\Omega\rangle$$

$$\Psi_k^\dagger\Psi_l^\dagger|\Omega\rangle$$

$$\vdots$$

$$\Psi_1^\dagger\dots\Psi_n^\dagger|\Omega\rangle$$

Representation of Braid Group

The action of this braiding on the Majorana zero modes is

$$\rho_i : \begin{cases} \gamma_i \to \gamma_{i+1} \\ \gamma_{i+1} \to -\gamma_i \\ \gamma_j \to \gamma_j & j \neq i, i+1 \end{cases}$$

We need to find representations satisfying $R_i \gamma_j R_i^\dagger = \rho_i(\gamma_j)$. One choice is

$$R_i = \exp\left(\frac{\pi}{4}\gamma_{i+1}\gamma_i\right) = \frac{1}{\sqrt{2}}\left(1 + \gamma_{i+1}\gamma_i\right)$$

Construction of Hadamard Gate

State: $|0\rangle = |\Omega\rangle$, $|1\rangle = \Psi_1^\dagger \Psi_2^\dagger |\Omega\rangle$

Representation:

$$R_1 = \frac{1}{\sqrt{2}}(1 + \mathrm{i} - 2\mathrm{i}\Psi_1^\dagger \Psi_1) = e^{\mathrm{i}\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{bmatrix}$$

$$R_3 = \frac{1}{\sqrt{2}}(1 + \mathrm{i} - 2\mathrm{i}\Psi_2^\dagger \Psi_2) = e^{\mathrm{i}\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{bmatrix}$$

$$R_2 = \frac{1}{\sqrt{2}}[1 - \mathrm{i}(\Psi_2 + \Psi_2^\dagger)(\Psi_1 - \Psi_1^\dagger)] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}$$

Notice that Hadamard gate is $H = R_1^{-1} R_2 R_1^{-1}$.

Construction of CNOT Gate

CNOT gate requires auxiliary Majorana zero modes. Auxiliary Majorana zero modes do not form qubits.

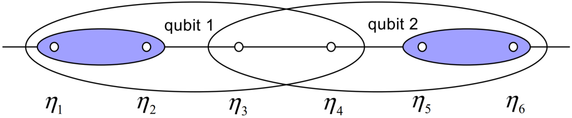

Figure: Georgiev L S, Phys Rev B, 2006, 74(23): 235112.

MZM 1 and 2 form qubit 1, MZM 5 and 6 form qubit 2.
MZM 3 and 4 are auxiliary MZMs. In total 6 MZMs are needed.

Construction of CNOT Gate

State: $|00\rangle = |\Omega\rangle, \quad |01\rangle = \phi^\dagger \Psi_2^\dagger |\Omega\rangle,$

$|11\rangle = \Psi_1^\dagger \Psi_2^\dagger |\Omega\rangle, \quad |10\rangle = \phi^\dagger \Psi_1^\dagger |\Omega\rangle$

Representation:

$$R_1 = \frac{1}{\sqrt{2}}(1 + \mathrm{i} - 2\mathrm{i}\Psi_1^\dagger \Psi_1)$$

$$R_3 = \frac{1}{\sqrt{2}}(1 + \mathrm{i} - 2\mathrm{i}\phi^\dagger \phi)$$

$$R_5 = \frac{1}{\sqrt{2}}(1 + \mathrm{i} - 2\mathrm{i}\Psi_2^\dagger \Psi_2)$$

$$R_4 = \frac{1}{\sqrt{2}}[1 - \mathrm{i}(\Psi_2 + \Psi_2^\dagger)(\phi - \phi^\dagger)]$$

Construction of CNOT Gate

$$R_1 = e^{\mathrm{i}\pi/4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix}, R_3 = e^{\mathrm{i}\pi/4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_5 = e^{\mathrm{i}\pi/4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix}, R_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 1 \end{bmatrix}$$

Notice that $\mathrm{CNOT} = R_3^{-1} R_4 R_3 R_1 R_5 R_4 R_3^{-1}$.

Summary

- Quasiholes (vortices) are non-abelian anyons. Electron-vortex bound states have Majorana zero modes.
- Many Majorana zero modes create topologically seperated ground states. Transition from different ground states can only be done by braiding quasiholes.
- Phase-shift gate cannot be performed in a topologically protected way. Universal quantum computation will require vulnerable operations. Software-level error corrections are needed.