Introduction to $\mathrm{U}(1)$ Gauge Field Theory and Its Quantization

Zhang Chang-kai

Department of Physics,
Beijing Normal University

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Definition of Topological Space

Definition from textbook

Definition

Topological space is a set X together with a subset $\mathcal{T}$ of its power set satisfying

1. $\varnothing, X \in \mathcal{T}$
2. Closed under finite intersection
3. Closed under arbitrary union

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Example

Open set of $\mathbb{R}^n$ is defined as

$$U = \mathrm{span}\{(a, b)\}$$

Construction of Manifold

Definition

A map f between topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{S})$ is continuous if

$$(\forall V \subset \operatorname{img}(f) \in \mathcal{S}) f^{-1}[V] \in \mathcal{T}$$

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(n-dimensional C^r) manifold is a topological space with open cover U_α satisfying

1. There exists homeomorphism $\psi_\alpha : U_\alpha \to V_\alpha \quad (\forall U_\alpha)$
2. For $U_\alpha \cap U_\beta \neq \varnothing$, composite $\psi_\beta \circ \psi_\alpha^{-1}$ is C^r.

Lie Group

Definition

Group G is a set with multiplication $\cdot : G \times G \to G$ satisfying

1. $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
2. $(\exists e) \quad ge = eg = g$
3. $(\forall g, \exists g^{-1}) \quad gg^{-1} = g^{-1} g = e$

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Left transformation L_g is a map $L_g : h \mapsto gh$.

Lie Algebra

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Vector field $\bar{A}$ is left invariant if $L_{g*}\bar{A} = \bar{A}$.

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Lie bracket of vector space V is a map $V \times V \to V$ satisfying

1. $[A, B] = -[B, A]$
2. $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$

A vector space with Lie bracket forms a Lie Algebra.

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Definition

Define

$$[A, B] = [\bar{A}, \bar{B}]_e$$

with commutator of vectors in Lie Group G. This forms the Lie Algebra of Lie Group G.

Principal Bundle

Definition

Left (right) action of manifold K is a map $L(R) : K \times G \to K$ satisfying

1. $L(R)_g$ is a diffeomorphism.
2. $L_{gh} = L_g L_h$ and $R_{gh} = R_h R_g$

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Definition

Principal bundle is constructed with a bundle manifold P, a base manifold M and a structure group G, satisfying

1. G has a free right action on P
2. Exists a smooth onto projection map $\pi : P \to M$ satisfying $\pi^{-1}[\pi[p]] = \{pg, g \in G\}$
3. Exists a local trivialization $T_U : p \mapsto (\pi(p), S_U(p)) \in U \times G$

Associated Bundle

Definition

Fibre bundle associated to principal bundle P is a set $P \times F/\sim$ with

1. Left action $\chi_g(f) = gf$
2. Induced free right action $\xi_g(p, f) = (pg, g^{-1}f)$
3. Equivalence relation $(p, f) \sim (pg, g^{-1}f)$

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Definition

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Connection is a smooth $\mathfrak{g}$-valued 1-form ω_U for each local trivialization T_U. And if the transition map of T_U and T_V is g_{uv}, there is

$$\omega_V = g_{uv}^{-1}\omega_U g_{uv} + g_{uv}^{-1}\mathrm{d}g_{uv}$$

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Definition

Curvature of connection is defined as

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

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4-dimensional connected Hausdorff second-countable orientable time-orientable framed smooth manifold

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Gauge field strength is the curvature of principal bundle

$$\Omega_{\mu\nu} = eF_{\mu\nu}$$

First Quantization

Definition

First quantization is to construct the Lagrangian of matter field as

$$\mathcal{L} = -\mathrm{i}\bar{\psi}(\gamma^\mu \nabla_\mu + m)\psi$$

where ∇_μ is the covariant derivative of associated bundle

$$\nabla_\mu = \partial_\mu + \omega_\mu$$

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Definition

Construct the kinematic term as

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu} \cdot F^{\mu\nu})$$

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Matter term

$$\mathcal{L} = -\mathrm{i}\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi$$

Gauge term

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu} \cdot F^{\mu\nu})$$

Interaction term

$$\mathcal{L} = -\mathrm{i}e\bar{\psi}\gamma^\mu A_\mu \psi$$

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Path Integral of Gauge Field

The original path integral should be

$$W[J] = \int [\mathcal{D}\omega] \exp\{\mathrm{i}\int \varepsilon(-\frac{1}{4}F_{\mu\nu}\cdot F_{\mu\nu} + J^{\mu}\cdot\omega_{\mu})\}$$

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However, we find that this is uncertain since the gauge freedom makes the four components of connection not all independent. Notice that the electromagnetic wave is a transverse wave. Thus we introduce temporal gauge fixing

$$W[J]=\int[\mathcal{D}\omega\delta(\omega_0)]\exp\{\mathrm{i}\int\varepsilon(-\frac{1}{4}F_{\mu\nu}\cdot F_{\mu\nu}+J^{\mu}\cdot\omega_{\mu})\}$$

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However, this will destroy the Lorentz invariance.

General Gauge Fixing

Consider Lorentz invariant gauge fixing

$$f(\omega) - C = 0$$

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Remarks. The strictness of this determinant is still under investigation. Adding $\Sigma_f \Sigma_f^{-1}$ into the path integral and get

$$W[J] = \int [\mathcal{D}\omega \mathcal{D}g \delta(f(\omega) - C)] \Sigma_f \exp\{\mathrm{i} \int \varepsilon(-\frac{1}{4} F_{\mu\nu} \cdot F_{\mu\nu} + J^{\mu} \cdot \omega_{\mu})\}$$

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By adding a Gaussian factor of C

$$\int [\mathcal{D}C] \exp\{-\frac{\mathrm{i}}{2\alpha}\int \varepsilon C \cdot C\}$$

and integrate C, there will be

$$W[J] = \int [\mathcal{D}\omega] \exp\{\mathrm{i}\int \varepsilon(-\frac{1}{4}F_{\mu\nu} \cdot F_{\mu\nu} - \frac{1}{2\alpha} f \cdot f + J^{\mu} \cdot \omega_{\mu})\}$$

This is the desired result.

Thanks!