On the AdS/CFT Correspondence

Zhang Chang-kai

School of Physics and Astronomy, University of Manchester
Oxford Road, Manchester, M13 9PL, UK

E-mail: phy.zhangck@gmail.com

Supervised by Prof. Niels Walet

Started: September, 2017
Finished: October, 2017
Updated: November, 2017

Abstract.

This is a brief review of the AdS/CFT correspondence. The AdS/CFT correspondence has, since it is proposed in 1997, ignited an animated research program and drawn attention from various theoretical (and even experimental) societies. This extraordinary correspondence, later rephrased as gauge/gravity duality, suggests that there exists a correspondence between a gravitational theory and a conformal field theory in one lower dimension. Here, we will give a review of both the original derivation and a modern implication of the AdS/CFT correspondence with a revealing introduction of the concepts used. Also, we present several checks of this correspondence and explain how it can be applied in areas like computing the entanglement entropy.

Keywords: AdS/CFT Correspondence; Holography; Entanglement Entropy

1. Introduction

Gravity has, for a long period of time, been absent in quantum area, as it is too weak to exert any observable effects. Also, few theories other than string theory can provide quantum gravity in a consistent way. However, the situation changed dramatically since 1997 when Maldecena [1] proposed that there is a correspondence between a classical supergravity with an $\mathrm{AdS}_5 \times \mathrm{S}_5$ background and a super Yang-Mills conformal field theory. This is the start of the AdS/CFT correspondence which is now more generally known as gauge/gravity duality.

This correspondence was soon found extremely useful, since the weakest version of the correspondence suggests that one can use a classical supergravity to tackle a strongly coupled conformal field theory. Due to the lack of tools to study non-perturbative field theory, it is currently impossible to prove the correspondence. However, various tests in simpler situations have been made since the proposal of the duality and now physicists are highly convinced that the correspondence does exist. [2] Therefore, the questions now turn to the prospective application of the AdS/CFT correspondence.

One of the most crucial application is on the study of strongly coupled conformal field theory. It is now possible to compute quantities that are hard to compute in conventional ways using the gravity dual. The entanglement entropy, which measures the extent of the entanglement, is an example of these quantities. Under gravity picture, the entanglement entropy becomes the black hole entropy, which has already been investigated quite thoroughly.

In this dissertation, we provide a review of the original approach of the AdS/CFT correspondence, as well as a modern intuition coming from the holographic principle. Also, we illustrate how the correspondence can be applied in the computation of the entanglement entropy. For a broader and more general review of the AdS/CFT correspondence, refer to Hubeny(2014). [2] For technical details, refer to Erdmenger(2012) and Ramallo(2015). [3][4] Also, Aharoney(1999) [5] offers an in-depth introduction in related areas.

2. Concepts in String Theory

In contrast to other high energy theories whose fundamental object is the point particle, the string theory suggests that the fundamental object is a string and all fundamental particles can be depicted by the vibration of the string. Mathematically, the string is described by a world sheet $X^\mu(\tau, \sigma)$, where X^μ gives the spacetime coordinate, τ the proper time and σ the string coordinate. Conventionally, τ takes value from $\mathbb{R}$ and σ from $\mathbb{R}$ or $[0, \pi]$. Usually, the string has finite length. And its tension allows it to vibrate and excite different particles. For more details in string theory, consult Tong(2016). [6]

2.1. Quantization of Strings

A string can be expanded to Fourier series, and we gain the coefficient x^μ, p^μ and α_i^μ ($\tilde{\alpha}_i^\mu$) from the series. Then, similar to the quantization procedure in other quantum theories, we promote these coefficients to operators and ask them to satisfy certain commutation relations.

Just as in Quantum Field Theory, there is a zero-point energy during the quantization procedure. An appropriate analysis on the zero-point energy fixes the dimension of the spacetime where the string can live in. For string theory, this dimension is $D = 26$. It is reduced to $D = 10$ when adding supersymmetry.

2.2. Boundary Conditions of Strings

The equation of motion of the string requires appropriate boundary condition at the end point of the string, i.e. the condition for $X^\mu(\tau, 0)$ and $X^\mu(\tau, \pi)$. One possibility is the periodic boundary condition

$$\text{Periodic:} \quad X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{1}$$

This condition makes a closed string, as it repeats itself after the string coordinate travels a distance of 2π. We can also have open strings which does not satisfy the periodic condition.

For open strings, string coordinate σ is limited to $[0, \pi]$ and we have two possible boundary conditions — Neumann and Dirichlet boundary condition

$$\text{Dirichlet:} \quad \delta X^\mu|_{\sigma=0} = \delta X^\mu|_{\sigma=\pi} = 0 \tag{2}$$

$$\text{Neumann:} \quad \partial_\sigma X^\mu|_{\sigma=0} = \partial_\sigma X^\mu|_{\sigma=\pi} = 0 \tag{3}$$

where Dirichlet boundary condition will fix the end point of an open string to some value and Neumann boundary condition allows the end point to move freely.

2.3. Dirichlet Brane

Strings with end points fixed may look odd and less physical, especially when the time component is fixed. However, if we impose only some of the components of an open string under Dirichlet boundary condition and the other components under Neumann boundary condition, the string will end up in a subspace in the high dimensional spacetime.

Concretely, let X^μ, $\mu = 0, \ldots, p$ satisfy Neumann boundary condition while X^μ, $\mu = p+1, \ldots, D-1$ satisfy Dirichlet boundary condition. The end of the open string will only be able to move in a $p+1$ dimensional subspace of a D dimensional spacetime. This subspace is called Dirichlet p-brane, or Dp-brane. The dual descriptions of the Dp-brane are the starting point of the AdS/CFT correspondence.

2.4. String Spectrum

The coefficients α_i^μ ($\tilde{\alpha}_i^\mu$) of the Fourier series of strings become operators after string quantization. Thus, these operators can be used to generate the particle spectrum in string theory. As the AdS/CFT correspondence only cares about systems with low energy, only the first excited states are studied here.

For closed string, a mechanism called level matching during the quantization procedure requires operator α_i^μ and $\tilde{\alpha}_j^\nu$ to act at the same time. Therefore, the first excited state is

$$\tilde{\alpha}_{-1}^\mu \alpha_{-1}^\nu |\Omega\rangle \tag{4}$$

where $|\Omega\rangle$ is the string vacuum state (different from physical vacuum state). This excitation has two indices μ and ν which transform according to $\mathrm{SO}(1, D-1)$. Thus, these states can be identified as the quantum of a second order tensor field. Moreover, the representation theory indicates that this tensor can be decomposed into three fields

$$\text{Symmetric: } G_{\mu\nu}(X), \quad \text{Anti-symmetric: } B_{\mu\nu}(X), \quad \text{Scalar: } \Phi(X) \tag{5}$$

These fields are related to gravity, charge and string coupling respectively. And they are vital parts in the closed string description of Dp-brane system.

For open string, we have $\alpha_i^\mu = \pm\tilde{\alpha}_i^\mu$, where $+$ and $-$ for Neumann and Dirichlet boundary condition respectively. Therefore, we essentially have only one set of operators. Hence, the first excited state is $\alpha_i^\mu|\Omega\rangle$. However, as we have imposed different boundary conditions for different components, the symmetry group breaks from $\mathrm{SO}(1, D-1)$ into $\mathrm{SO}(1,p) \times \mathrm{SO}(p+1, D-1)$. Therefore, we can identify two fields from the excitation

$$\text{Longitudinal:} \quad \alpha_{-1}^{\mu}|\Omega\rangle, \quad \mu = 0, \ldots, p \tag{6}$$

$$\text{Transverse:} \quad \alpha_{-1}^{I}|\Omega\rangle, \quad I = p+1, \ldots, D-1 \tag{7}$$

The longitudinal component transforms under $\mathrm{SO}(1,p)$ and then is identified as the quantum of a vector field. The transverse component transforms as scalar under $\mathrm{SO}(1,p)$ and therefore identified as the quantum of scalar fields living on the brane.

3. Low Energy Effective Theory

The string theory works usually in high energy. Thus, to address the AdS/CFT correspondence, its low energy behaviour is needed. In low energy, only massless states are excited. Moreover, the scale invariance of string action reveals that the beta functions of the effective fields corresponding to these massless states should vanish, which functions as equations of motion and gives the effective actions.

Then, we study the system of a stack of N Dp-branes. This system has a dual description. [3][5] From the closed string point of view, it is a massive charged object which sources the various supergravity fields; and from the open string point of view, it is the hypersurface where open string ends.

3.1. Weyl Invariance

There are two effectively identical formulations of string action — Nambu-Goto action and Polyakov action [6]

$$\text{Nambu-Goto action:} \quad S_{\mathrm{NG}} = \int \mathrm{d}^2\sigma \sqrt{-\det(\partial_a X_\mu \partial_b X^\mu)} \tag{8}$$

$$\text{Polyakov action:} \quad S_{\mathrm{P}} = \int \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \partial_a X_\mu \partial_b X^\mu \tag{9}$$

The Polyakov action gets rid of the square root and makes the quantization easier. However, the Nambu-Goto action preserves the symmetry under string coordinate transformation, which turns into the three unphysical degrees of freedom in h_{ab}. Therefore, three gauge symmetries are called to cancel these degrees of freedom.

An ostensible choice is the diffeomorphism symmetry, which cancels two of them. And the rest one will be eliminated by the Weyl invariance. But Weyl invariance implies that the theory is scale invariant. And the very quantity in Quantum Field Theory which depicts how theory varies according to the energy scale is the beta function. Therefore, in the low energy limit, the beta function of fields $G_{\mu\nu}$, $B_{\mu\nu}$ etc. should vanish.

3.2. Closed String Description

In closed string description, we have the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Also, in superstring theory, it is possible to have extended fields representing extra charges (called Ramond-Ramond charge), as well as fermionic partners to preserve supersymmetry. Therefore, the effective action can be written as [3]

$$S = \frac{1}{2\kappa}\int \mathrm{d}^{10}x\sqrt{-G}e^{-2\Phi}(\mathcal{R} - \frac{1}{2}|H_3|^2 + 4\partial_\mu\Phi\partial^\mu\Phi) + S_{\text{extra}} \tag{10}$$

where κ is the supergravity coupling constant, $\mathcal{R}$ the Ricci tensor of the metric $G_{\mu\nu}$ and $H_3 = \mathrm{d}B_2$ the field strength (subscript indicates its order). The S_{extra} term carries all the contributions from the charges and fermions.

The solutions to the equations of motion derived from the action above contain black holes. Particularly, the classical solution with a black hole in $p+1$ dimensions is called a black p-brane. Strominger(1991) [7] provides a general form of the black p-brane solution

$$\mathrm{d}s^2 = \frac{1}{\sqrt{H(r)}}\mathrm{d}x_\mu\mathrm{d}x^\mu + \sqrt{H(r)}\mathrm{d}y_I\mathrm{d}y^I, \quad H(r) = 1 + \frac{R^{7-p}}{r^{7-p}} \tag{11}$$

where $\mu = 0, ..., p$ and $I = p+1, ..., 9$; $r = \sqrt{y_I y^I}$ and R is a parameter related to N whose significance will become clear in the later context.

The black p-brane and Dp-brane share many interesting similarities [5], especially the Ramond-Ramond charges. This draws Polchinski(1995) [8] to conclude that the two objects are just two descriptions of the same physics. This insight motivates the AdS/CFT correspondence.

3.3. Open String Description

The open string with ends in a Dp-brane gives rise to a vector field A^a and several scalar fields ϕ^I, which describes the dynamics of the brane. Again, to address the low energy effective theory, we require the beta function to vanish. The effective action of equation $\beta(F_{\mu\nu}) = 0$ is the Born-Infeld action which has been studied very early. [9] Together with the form of Dirac action which describes the brane, we get the Dirac-Born-Infeld action (or DBI action) [3][6]

$$S = -T_p\int \mathrm{d}\xi\sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \tag{12}$$

where T_p is the Dp-brane tension, ξ a set of brane coordinates, $\gamma_{ab} = \partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu}$ the pull-back metric on the brane, α' the inverse string tension and F_{ab} the field strength of the vector field. If set $X^\mu = \xi^\mu$ for $\mu = 0, ..., p$, then

$$\gamma_{ab} = \eta_{ab} + \frac{\partial X^I}{\partial\xi^a}\frac{\partial X^J}{\partial\xi^b}\delta_{IJ} \tag{13}$$

where component X^I originates the scalar fields ϕ^I. Thus, when α' is sufficiently small, we are able to expand the DBI action and obtain

$$S = -(2\pi\alpha')T_p\int \mathrm{d}\xi\left[\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^I\partial_b\phi_I + \cdots\right] \tag{14}$$

where $\phi^I = X^I/2\pi\alpha'$. All the terms coming with higher order in α' are then suppressed. Hence, we see that in small α', DBI action depicts a Maxwell theory coupled with massless scalar fields.

The above discussion is based on a single Dp-brane. Now, suppose there are N coincident Dp-branes. The open strings can then choose to start and end from any Dp-branes. Therefore, we have N choices for both the start point and the end point, giving N^2 different strings and thus ϕ^I fields.

These N^2 fields can be grouped together to form $(\phi^I)^m_n$ whose corresponding string starts at brane m and ends at brane n. A similar method can be applied to the A^a field. An analysis on the symmetries [6] reveals that the modification turns (14) into non-Abelian situation.

Besides, the brane system can also be coupled with closed string excitations $G_{\mu\nu}$ etc., and carry Ramond-Ramond charges. Therefore, a general form of the effective action can be written as

$$S = -T_p \int \mathrm{d}\xi \sqrt{-\det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + S_{\text{CS}} + S_{\text{bulk}} \tag{15}$$

where $g_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$, S_{CS} a Chern-Simons term which describes the interaction with Ramond-Ramond fields and S_{bulk} the supergravity propagating in the bulk.

4. The AdS/CFT Correspondence

Now, we have both the closed string and open string description of a system consisting of N Dp-branes.‡ To elucidate the AdS/CFT correspondence, consider the energy scale much lower than the string energy scale $1/\sqrt{\alpha'}$ (or simply send $\alpha' \to 0$) and see what the two descriptions can tell us. As the AdS/CFT correspondence concerns five-dimensional AdS space in and four-dimensional conformal field theory, we particularly care about D3-brane.

On the closed string side, the system is described by the background metric shown in equation (11) where now $p = 3$ and

$$R^4 = 4\pi\alpha'^2 g_s N \tag{16}$$

Under the limit $\alpha' \to 0$, the target spacetime will be separated into two regions. Consider the region where $r \to \infty$ ($r \gg R$) and the closed strings propagating to this region. From equation (11), the time component of the metric reads

$$g_{tt} = -H^{-1/2} = -(1 + R^4/r^4)^{-1/2} \tag{17}$$

Therefore, the energy observed at infinity E will tend to be infinitely redshifted, as it is related to that E_p measured at constant r by $E = \sqrt{-g_{tt}}E_p$. This makes the closed strings propagating in the bulk less able to interact with the near horizon region ($r \to 0$), since the absorption cross section goes like E^3R^8. [5] Moreover, the bulk metric turns free as $r \to \infty$ and $\alpha' \to 0$. Therefore, the bulk region becomes free supergravity in the limit.

In the near horizon region $r \to 0$ (or $r \ll R$), the closed strings will find them trapped and hard to escape. Also, when $r \ll R$, the metric becomes

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\mathrm{d}x^\mu \mathrm{d}x_\mu + \frac{R^2}{r^2}\mathrm{d}r^2 + R^2 \mathrm{d}\Omega_5^2 \tag{18}$$

‡ Here, N is usually large so that string coupling g_s can be small to suppress some quantum corrections.

where Ω_5 denotes the solid angle of the five-dimensional sphere. Recall that this geometry is $\mathrm{AdS}_5 \times \mathrm{S}_5$, a five-dimensional AdS space and a five-dimensional sphere. Hence, the near horizon region is described by a supergravity under $\mathrm{AdS}_5 \times \mathrm{S}_5$ background.

On the open string side, the D3-brane dynamics is governed by the excitation of the open string. The total effective action can be written as a combination of bulk, brane and bulk-brane interaction

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \tag{19}$$

The bulk-brane interaction can be seen from the coupling with $G_{\mu\nu}$ in the DBI action. Notice that the interaction coupling constant is $\kappa \sim g_s \alpha'^2$. [3][5] Therefore, as $\alpha' \to 0$, the interaction term vanishes, and we again have the system decoupled into a separate bulk and brane region.

In the bulk region, there lives a supergravity, whose action can be expanded schematically according to $g = \eta + \kappa h$ as [3][5]

$$S_{\text{bulk}} = \frac{1}{2\kappa} \int \sqrt{-g}\mathcal{R} \sim \int (\partial h)^2 + \kappa (\partial h)^2 h + O(\kappa^2) \tag{20}$$

As $\alpha' \to 0$, all the terms with κ are suppressed, which makes the bulk supergravity free. Therefore, the bulk also carries a free supergravity in the open string description.

In the brane region, the dynamics is provided by the DBI action. As we have argued in the previous section, the expansion of DBI action under small α' leads to a Yang-Mills field theory. A closer analysis on the supersymmetry in this theory [5] reveals that it is in fact an $\mathcal{N} = 4$ super Yang-Mills field theory with $\mathrm{SU}(N)$ gauge group in four dimensions, where $\mathcal{N} = 4$ indicates four copies of supersymmetry. It is known that this is conformally invariant. Therefore, there is a conformal field theory living on the brane.

In summary, from both descriptions, the system decouples into two parts. On the closed string side, they are free supergravity in the bulk and supergravity under $\mathrm{AdS}_5 \times \mathrm{S}_5$ background near the horizon. On the open string side, they are free supergravity in the bulk and a conformal field theory on the brane.

Since the two views are just two descriptions of the same system, and we both have a free supergravity in the bulk, it is then natural to conjecture that their interpretation about the brane should be dual to each other. This leads to the AdS/CFT correspondence

> A supergravity on $\mathrm{AdS}_5 \times \mathrm{S}_5$ background is dual to a $D = 4$, $\mathcal{N} = 4$, $\mathrm{SU}(N)$ super Yang-Mills gauge theory.

A direct check of the correspondence is the symmetries in both sides. In super Yang-Mills theory, conformal symmetry provides an $\mathrm{SO}(2, 4)$ group. Also, the R-symmetry gives an $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$, where $\simeq$ essentially means the former is the double cover of the latter.

In supergravity on $\mathrm{AdS}_5 \times \mathrm{S}_5$, the isometry of AdS_5 gives the $\mathrm{SO}(2, 4)$ and rotational symmetry of S_5 yields the $\mathrm{SO}(6)$ symmetry. Therefore, the symmetries of the two descriptions match with each other.

4.1. Discussion on Parameters

The duality also suggests a correspondence of parameters from both descriptions. This can be achieved by comparing the coefficient of the F^2 term in the super Yang-Mills theory. A typical coefficient will be $1/g_{\mathrm{YM}}^2$ with g_{YM} being the coupling constant. And the expansion of DBI action yields an $1/g_s$ with g_s being the string coupling before the F^2 term. Hence, we should have

$$g_{\mathrm{YM}}^2 \equiv g_s \tag{21}$$

where $\equiv$ indicates that there may be a flexibility regarding a constant factor.

Next, we analyze the applicability of the two corresponding theories. In super Yang-Mills theory, there is perturbative expansion with respect to the t'Hooft coupling constant

$$\lambda_{\mathrm{tH}} = g_{\mathrm{YM}}^2 N \equiv g_s N \tag{22}$$

where the last equality utilizes the correspondence of the parameters. To validate the perturbative theory, the coupling should be sufficiently small. Therefore, the super Yang-Mills theory is applicable when λ_{tH} is small.

On the other hand, we have the background metric of $\mathrm{AdS}_5 \times \mathrm{S}_5$. This is a classical solution and thus can only be trusted when the gravitation is sufficiently weak. The strength of gravity can be quantified by the scalar curvature which goes like

$$\mathcal{R} \propto R^{-2} \propto \lambda_{\mathrm{tH}}^{-1/2} \tag{23}$$

The gravity is weak if and only if the curvature is small, which leads to a large λ_{tH}. Therefore, the supergravity description is valid when λ_{tH} is large. And we find that the two description works in exactly an opposite situation.

4.2. Weak and Strong Correspondence

This result motivates two versions of the correspondence. [3] The weak version suggests that the correspondence is valid when λ_{tH} is large, as the super Yang-Mills theory should be valid for all coupling, even though the perturbation theory will fail in this case. In this scenario, we have a classical supergravity on one side, which has been well understood; and a strongly coupled gauge theory on the other, which we currently have no tools to study. Therefore, the weak version of the correspondence demonstrates that a gravitational theory can be used to tackle a strongly coupled gauge theory.

A strong version of the correspondence suggests that the duality also appears even when λ_{tH} is small. In this case, the super Yang-Mills theory is weakly coupled and the perturbation theory works. On the other hand, the gravity becomes strong and the quantum effects are not negligible. Therefore, the strong version insinuates that a weakly coupled gauge theory can be used to address some quantum gravity.

Both versions of the correspondence have profound applications, either in the strongly correlated system or the quantum gravity effects. It is this trait that makes the AdS/CFT correspondence more and more significant nowadays.

5. Intuition from Holography

The previous arguments on the AdS/CFT correspondence follow directly the Maldecena's original derivation, which is based highly on the string theory. However, we can also find some hints of the duality without invoking the string theory. The holographic principle [10][11] initiated by the study of black hole entropy propounds a direct and intuitive approach to the correspondence. And one can even recover some structures in string theory with this approach. [2][12]

The holographic principle originates from the unusual property of black hole entropy. Ordinary objects have an entropy proportional to the volume. Howbeit, the black hole entropy is proportional to the horizon area [13]

$$S_{\mathrm{BH}} = \frac{1}{4}A \tag{24}$$

This indicates that information in a gravitational system may be encoded in its boundary with one lower dimension, which becomes the central idea for the holographic approach to the AdS/CFT correspondence. [12]

Concretely, the story starts from the implication of representation theory that a spin-2 graviton can be a composition of two spin-1 gauge bosons. However, the no-go theorem eradicates this possibility as long as the graviton lives in the same dimension as its constituent gauge bosons. This is exactly when holographic principle comes to help, which suggests that the graviton may propagate in one higher dimension, and thus surpasses the no-go theorem.

Next, consider a gauge field theory with the following requirements: large N, to provide enough degrees of freedom for an extra dimension; strong coupling, as there are no signs of gravitation in weak coupling; $\mathcal{N}=4$ supersymmetry, to provide stability and conformal invariance. The conformal invariance makes the energy scale a desirable choice as a fifth dimension, as the vanishing beta function allows the coupling to remain strong for a wide range of energy scale.

Then, we seek for a five-dimensional geometry with the same spacetime symmetry as the field theory proposed above — the Poincaré and conformal symmetry. The most general one will be the AdS_5 spacetime:

$$\mathrm{d}s^2 = \frac{R^2}{z^2}(\mathrm{d}x_\mu \mathrm{d}x^\nu + \mathrm{d}z^2) \tag{25}$$

The AdS_5 spacetime preserves an isometry $z \to \lambda z, x^\mu \to \lambda x^\mu$ where the latter one becomes the dilation transformation at the four-dimensional boundary of the conformal field theory and simultaneously results in a rescale of energy $E \to E/\lambda$. [14] This implies that coordinate z should be inversely proportional to the energy scale. And indeed, a trivial change of variable $z = R^2/r$ will recast (25) to the form (18) seen previously.

But one can go even further. [12] If we also consider the supersymmetry in the gravity side, the full extension will eventually point to the supergravity on $\mathrm{AdS}_5 \times \mathrm{S}_5$ which requires a ten-dimensional target spacetime. Then, the field theory can be extended according to the dimension and produces the scalar fields. Thus, step by step, we may even *uncover* the string theory structures once assume that the duality holds.

6. Application in Entanglement Entropy

Numerous applications of the AdS/CFT correspondence have been found ever since its proposal, and the holographic computation of entanglement entropy is amid one of them. Conventionally, the entanglement entropy can be attained in two-dimensional quantum field theory, and nonetheless meet great obstacles in generalization into higher dimensions. And this is exactly when the dual gravitational theory proffers to help.

Hither in this section, we will present the idea of the entanglement entropy computation in higher dimensions using the AdS/CFT correspondence. A detailed review on this topic can be found in Nishioka(2009). [15]

6.1. Concept of Entanglement Entropy

The concept of entanglement entropy arises in the density matrix formalism of Quantum Mechanics. In this formalism, the quantum state is determined by the density matrix ρ. Two types of state can be identified through in the density matrix formalism — pure state and mixed state. All states that can be expressed by a state vector $|\psi\rangle$ in the Hilbert space will be a pure state, with density matrix being $\rho = |\psi\rangle\langle\psi|$; while mixed state can be represented by a probabilistic linear combination of the density matrix of the pure states which cannot be decomposed into the product of the pure states.

The pure and mixed states exhibit a critical property: $\rho^2 = \rho$ for pure state and $\rho^2 \neq \rho$ for mixed state. This property is dominant in determining the extent of entanglement in a system.

Now, consider a total system depicted by the density matrix ρ_{tot}, and can be (artificially) divided into two subsystems labelled by A and B. Then, suppose we want to retrieve some information about subsystem A. A straightforward idea is to trace out (or path integrate out) the degrees of freedom of subsystem B, which gives rise to the concept of reduced density matrix

$$\rho_A = \mathrm{tr}_B\, \rho_{\text{tot}} \tag{26}$$

Next, if the subsystem A and B are not entangled with each other, it is manifest that the total density matrix takes the form $\rho_{\text{tot}} = \rho'_A \otimes \rho'_B$ where $\otimes$ denotes tensor product and ρ'_A, ρ'_B are pure states in subsystem A, B respectively. In this case, a direct computation yields

$$\rho_A = \mathrm{tr}_B\, \rho'_A \otimes \rho'_B = \rho'_A \tag{27}$$

where the left hand side is the reduced density matrix of subsystem A. Therefore, we see that without entanglement, the reduced density matrix will turn out to be pure.

On the other hand, if the subsystem A and B are entangled with each other, the total density matrix will then be impossible to be decomposed into the tensor product of pure states.

In this case, the reduction of total density matrix will produce a mixed state. Consequently, the extent of mixture becomes an indicator of the extent of entanglement.

Following the above idea and recall that the von Neumann entropy takes the form of $-\rho\log\rho$, we define the entanglement entropy (of subsystem A) by

$$S_A = -\mathrm{tr}\,\rho_A \log \rho_A \tag{28}$$

From the expression, we see that if ρ_A is pure, then it is idempotent and thus makes the entanglement entropy vanish; while a mixed ρ_A will produce a positive value for the entropy, which matches our previous expectation.

6.2. Computation of Entanglement Entropy

Preliminarily, the entanglement entropy is calculated using techniques in Quantum Field Theory. Concretely, consider a $D{+}1$-dimensional manifold divided by two submanifolds A and B. And the entanglement entropy defined by (28) is computed using a *replica trick* as follows [15]

$$S_A = -\left.\frac{\partial}{\partial n}\mathrm{tr}\,\rho_A^n\right|_{n=1} = -\left.\frac{\partial}{\partial n}\log\mathrm{tr}\,\rho_A^n\right|_{n=1} \tag{29}$$

where the last equality is gained noticing $\mathrm{tr}\,\rho_A = 1$. Hence, the task now is to evaluate $\mathrm{tr}\rho_A^n$ for a given field theory.

A conventional method of evaluation is to use path integrals. Observe that the index of the density matrix in a field theory becomes the field configuration. Therefore, we need to duplicate the total manifold to get n copies of spacetime and thus n number of ρ_As. Then take the trace (now essentially path integral) successively. Hence, we are actually performing path integral in an n-sheeted spacetime, which can be nearly implausible unless the spacetime is two-dimensional.

Consequently, the power of the AdS/CFT correspondence emerges. For dimensions more than two, the entanglement entropy can be computed using the black hole entropy formula in one higher dimension. Concretely, the AdS/CFT correspondence posits the following equality in partition function

$$Z_{\text{AdS}} = Z_{\text{CFT}} \tag{30}$$

Notice that there is a relationship between the entropy and partition function as $S \sim \log Z$. Then, quantity $\mathrm{tr}\rho_A^n$ plays the role as the partition function. Therefore, the target now switches to the computation of the partition function in a $D{+}2$-dimensional AdS spacetime.

The system A is now a D-dimensional surface in the AdS spacetime, and our task next is to find a surface with the same boundary ∂A that acts as the holographic screen of the system. This surface, denoted by γ_A, is proposed to be the static minimal surface in the AdS spacetime with boundary ∂A. [16] This can be achieved by finding a surface which makes the variation of area functional vanish. Going back to the two-dimensional case, this surface becomes the geodesics in AdS_3 spacetime.

Given the surface γ_A, we can now construct the action of the gravity. The action takes the usual form of Einstein-Hilbert action. And we need to maintain that the Ricci scalar takes the form [15]

$$\mathcal{R} = 4\pi(1-n)\delta(\gamma_A) + \mathcal{R}^{(0)} \tag{31}$$

where $\mathcal{R}^{(0)}$ is the Ricci scalar of pure AdS spacetime. In the case of AdS_3, the above scalar curvature represents a spacetime with a deficit angle $2\pi(1-n)$ near γ_A, which fits with the field theory case where we have an n-sheeted Riemann surface. However, there are currently no derivations in other dimensions. Therefore, we have to assume that the above formulation holds for an arbitrary spacetime dimension.

This assumption engenders the Einstein-Hilbert action

$$S_{\text{EH}} \sim -\log Z_{\text{AdS}} = -\frac{1}{16\pi G_N^{(D+2)}} \int \mathcal{R} + \cdots \simeq -\frac{(1-n)\text{Area}(\gamma_A)}{4\pi G_N^{(D+2)}} \tag{32}$$

where $G_N^{(D+2)}$ is the $D+2$-dimensional Newton constant of AdS gravity. Therefore, we can holographically derive the entanglement entropy as

$$S_A = -\frac{\partial}{\partial n} \log \operatorname{tr} \rho_A^n \bigg|_{n=1} = -\frac{\partial}{\partial n} \log Z_{\text{CFT}} \bigg|_{n=1} \tag{33}$$

$$= -\frac{\partial}{\partial n} \log Z_{\text{AdS}} \bigg|_{n=1} = -\frac{\partial}{\partial n} \frac{(1-n)\text{Area}(\gamma_A)}{4\pi G_N^{(D+2)}} \Bigg|_{n=1} = \frac{\text{Area}(\gamma_A)}{4\pi G_N^{(D+2)}} \tag{34}$$

where we have used the AdS/CFT correspondence to progress from the first to the second line. The above formula indicates that the entropy is proportional to the surface area of γ_A, which is analogous to the Bekenstein-Hawking entropy of black hole proportional to the area of the horizon.

6.3. Checks of the Correspondence

A direct way to check formula (34) is to apply it in the two-dimensional case where we already have the analytical result from the field theory. In the two-dimensional situation, Area(γ_A) becomes the length of the geodesics with the given start and end point. Ryu&Takayanagi(2006) [16] pointed out that the two results do coincide.

In higher dimensions, there are no known methods to provide analytical checks. Howbeit, we can still perform numerical computations in field theory and compare with the result given by holographic approach. Hence, consider an infinite strip in four dimensions and compute its entanglement entropy numerically [16][17]

$$S_A^{\text{AdS}} \approx -0.0510 \cdot u \tag{35}$$

$$S_A^{\text{CFT}} \approx -0.078 \cdot u \tag{36}$$

where u is some common factor containing relevant parameters in both theories. And we have omitted in both cases a common term involving the ultraviolet cutoff. It is discovered that the two results exhibit the same magnitude, but with a difference in numbers. This aberration is, however, within anticipation, since the holographic figure holds under strong coupling limit while the field theory estimation is done in the free circumstance.

7. Remarks on Other Applications

The actual application of the AdS/CFT correspondence can be considerably diverse, and the utilization in the entanglement entropy is a notable one of them. One of the profound features of the correspondence is its weak/strong coupling duality, which elicits in numerous programs in studying strongly coupled system using its gravitational dual. [2] Examples include the AdS/QCD which concerns strong interactions and the AdS/CMT regarding the strongly correlated systems in condensed matter theory.

On the other hand, as an inverse practice on the AdS/CFT correspondence, we can tackle some quantum gravity issues. The recent idea ER=EPR [18] suggests that two entangled black holes (EPR) are connected through the interior via wormholes, or Einstein-Rosen bridge (ER), which provides unique insights towards quantum behaviours of black holes and engenders possible solutions to the firewall paradox. More are now on the way to be revealed.

8. Conclusion

The AdS/CFT correspondence arose when the dual interpretation of the Dp-brane was found. The Dp-brane is originally a subspace of the target space where open strings end. However, as open strings can join together to form closed strings and propagate out, the Dp-brane then becomes the source of the closed strings, which brings about the closed string description. Therefore, there are two coincident descriptions of a system consisting of N Dp-branes. The behaviour of these two descriptions in low energy motivates the correspondence.

Under low energy limit, or $\alpha' \to 0$, the physics can be described by low energy effective theory. As the energy drops below the decoupling limit, both descriptions give a free supergravity in the bulk, and a conformal field theory on one side and a supergravity on $\mathrm{AdS}_5 \times \mathrm{S}_5$ background on the other. Since the two pictures aim at the same system, we conclude that there is a correspondence between the conformal field theory and the supergravity on $\mathrm{AdS}_5 \times \mathrm{S}_5$ background.

The original deduction of the AdS/CFT correspondence is contingent highly on the string theory. However, it is also possible to approach the correspondence via the holographic principle. The representation theory implies that the graviton can be composed of two gauge bosons, but the no-go theorem requires the two sides in different spacetime dimensions. Therefore, following the holographic principle, the gravitational theory should live in one higher dimension. Detailed investigation on the symmetries and other parameters also yield the AdS/CFT correspondence.

One of the representative application of the AdS/CFT correspondence is the holographic method of computing the entanglement entropy. The entanglement entropy measures the extent of the entanglement of a system and can be hard to reach for a high dimensional system. The correspondence suggests a way of computation in the gravitational dual of the field theory, which can be applicable universally.

The fascinating feature of the AdS/CFT correspondence that interconnects a gravitational theory with a non-gravitational one with antithetic coupling strength has triggered profound insights toward abundant areas in Physics. However, we are still away from the apex. More intriguing aspects in the strongly correlated systems are still waiting for investigations. More fundamental structures between quantum information and geometry are yet to be discovered. Consequently, our minds should be kept open to anticipate further revolutions empowered by the AdS/CFT correspondence.

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