Constructive Physics - Kernel

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ABSTRACT: This document presents a self-contained analytical construction of classical gauge field theory. Among all physical theories, the classical gauge field theory is the most inclusive framework for interaction theories. Also, it is the most fundamental theory with rigorous analytical support. Accordingly, a skeleton of this analytical structure with minimal mathematical concepts involved is hither introduced. The document consists of nine chapters with the first eight chapters being the mathematical preparation from logic to fibre bundle and the last chapter their application to the classical gauge field theory. As a self-contained construction, all the concepts mentioned in a definition should be found a definition in the previous context.

Contents

Methodology

This part introduces the methodology of studying Physics recognized by this document.

Definition of Physics

Definition. ***Physics*** *is the search for and application of rules that can help us understand and predict the world.*

This definition is formulated by

$$\bigcup_{i=1}^{m}\{\psi_i, \partial_\mu\psi_i\} \Rightarrow \mathcal{I}[\psi, \partial_\mu\psi] \mid + \sum_{i=1}^{n}\sigma_i \Rightarrow \mathcal{H}\Big|_{\wp}$$

All physical contents in this document is identified through this definition.

First Principle

The first principle of Physics is the symmetric principle

Symmetric Principle. *The* ***Symmetric Principle*** *is formulated as*

$$\imath\delta\mathcal{I} = 0$$

where $\mathcal{I}$ is the action and $\imath\delta$ is the internal variation.

Locality Condition

The locality condition is interpreted as

Locality Condition. *The action type-$\mathcal{I}$ has the formulation*

$$\mathcal{I} = \int \varepsilon\mathcal{L}$$

where Lagrangian $\mathcal{L}$ is a local continuous functional of ψ and $\partial_\mu\psi$ only.

First Theorem

The first theorem of Physics is the Noether's theorem

Noether's Theorem. *Every continuous symmetry in a theory $\pounds_\xi\psi$ corresponds to a conserved current*

$$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\pounds_\xi\psi - \ell^\mu$$

Noether's theorem is logically dependent on the symmetric principle.

Conventions and Notations

This section is to claim the default values of conventions and notations.

Abstract Index Notation This document complies abstract index notation.

Einstein's Summation Convention Repeating upper and lower indexes represents summation or contraction.

Indexes This document Greek letters represent abstract indexes and Latin letters represent specific indexes.

Fonts Latin letters in Italic represent variables and Latin letters in Roman represent particular specified meaning.

Equations Only key equations are numbered.

Unit System This document has two unit system — natural unit system and geometric unit system. Geometric unit system in this document is interpreted as $c = \hbar = 1$ and gravitational constant G is used as coupling constant.

Definition and Proof

This section illustrates the *undefined* mathematical grammar.

The definition should comply the following pattern

Definition. *A* ***concept*** *is a concept satisfying conditions.*

Definition. *A* ***concept*** *is adjective if conditions*

In the first formula, *concept* is the concept already well defined, **concept** is the concept that is defined through this definition. The second formula gives the definition of a concept as *adjective* **concept**. Only concepts and properties of concept should be expressed as definition.

The proof should comply the following pattern.

Proof. *condition* $\Rightarrow$ key point 1 $\Rightarrow \cdots \Rightarrow$ key point $i \Rightarrow \cdots \Rightarrow$ key point $n \Rightarrow$ *conclusion* $\square$

1 Foundation of Mathematics

Mathematics is the language of choice for scientific description and modelling. Mathematics describes objects in a logical way, preventing any mistakes due to inconsistency. A mathematical consistent theory is only mathematically falsifiable, which provides great property for applicative analysis.

1.1 Basics of Mathematics

This section illustrates the foundation of Mathematics.

1.1.1 Mathematical Logic

Mathematics is a kind of scientific language. Just as all the languages, Mathematics has to start with some concepts that are not able to be explained by itself. These concepts are called **intuitive** concepts. The intuitive concepts are listed as follows

Concept. ***Symbol*** *is the character in* $\LaTeX\,2_\varepsilon$

Concept. ***Independent Variable*** *is a symbol, conventionally being Greek letter.*

Note. In foundation of Mathematics, independent variables are usually *assigned* with **true** or **false**.

Note. A **predicate** is an independent variable *assigned* with true or false.

Concept. ***Predicate Constant*** $\in$ *is a symbol.*

Note. The independent variables and predicate constant can form a predicate as

$$\varphi \in \psi$$

Concept. ***Negation*** $\neg$ *is a symbol defined as*

ψ	true	false
$\neg\psi$	false	true

Concept. ***Disjunction*** $\vee$ *is a symbol defined as*

ψ	true	true	false	false
φ	true	false	true	false
$\psi \vee \varphi$	true	true	true	false

Concept. ***Conjunction*** $\wedge$ *is a symbol defined as*

ψ	true	true	false	false
φ	true	false	true	false
$\psi \wedge \varphi$	true	false	false	false

Concept. ***Implication*** $\rightarrow$ *is a symbol defined as*

ψ	true	true	false	false
φ	true	false	true	false
$\psi \rightarrow \varphi$	true	false	true	true

Note. The **inference** $\Rightarrow$ is a true implication.

Note. The truth of a predicate can be influence by some other predicates. The predicate ψ under condition predicate x will be denoted by $\psi|_x$ and its truth can be different for different condition predicates.

Concept. ***Existential Quantification*** $\exists$ *is a symbol defined as*

$$\exists x\, \psi|_x \Leftrightarrow \bigvee_x \psi|_x$$

where $\bigvee$ *represents the disjunction of all conditioned predicates.*

Concept. ***Universal Quantification*** $\forall$ *is a symbol defined as*

$$\forall x\, \psi|_x \Leftrightarrow \bigwedge_x \psi|_x$$

where $\bigwedge$ *represents the conjunction of all conditioned predicates.*

Concept. ***Priority*** $(\,), [\,]$ *is a pair of symbols. The independent variables and predicates inside priority are regarded as an independent variable or a predicate.*

Note. Priority ( ) is higher than [ ] unless stated.

Concept. ***Class*** C *is a symbol defined as*

$$C = \{x | \psi(x)\}$$

where x *is the independent variable and* $\psi(x)$ *is a predicate. The above expression means*

$$x \in C \Leftrightarrow \psi(x) \text{ is true}$$

Concept. ***Subclass*** S *of class* C *is a class satisfying*

$$x \in S \Rightarrow x \in C$$

Concept. ***Object*** x *is a symbol satisfying* $x \in C$ *where* C *is a class.*

Note. Class is an essential concept in mathematical logic. The definition of class will allow the existence of some weird class. For example, the Russell class is defined as

$$R := \{x \mid x \notin x\}$$

which means it "contains" only the object that does not belong to itself.

Note. Symbol $:=$ refers to "define as".

The introduction of intuitive concepts has ended here. All the concepts in the subsequent texts will have a rigorous definition.

1.1.2 Set Theory

This subsection illustrates the axiomatic set theory.

Note. Although this subsection aims at axiomatic set theory, the elaboration will not follow the genuine introduction of axiomatic set theory.

Note. Set is a very important concept in Mathematics. In mathematical logic, there is a concept, class. However, class allows some impractical situation like Russell class to exist. Thus a new concept is needed to make Mathematics pragmatic.

Definition. ***Set*** *is a class if it can be a subclass.*

Note. This definition can be different from many elaborations of axiomatic set theory which define the set through several axioms. In this document, the axioms will be presented as the definition of operations on sets.

Definition. ***Element*** *is the object of a set.*

Definition. ***Proper class*** *is a class that is not a set.*

Note. The Russell class is a proper class.

Definition. ***Subset*** *is the subclass of a set.*

Note. Statement "Set A is a subset of set B" is denoted by $A \subset B$ or $B \supset A$.

Definition. *Set A and set B is* ***equal*** *if*

$$A \subset B \text{ and } B \subset A$$

Definition. *Set A is a* ***proper subset*** *of set B, denoted by $A \subsetneq B$, if*

$$A \subset B \text{ and } A \neq B$$

Definition. ***Empty set*** *$\varnothing$ is a set that $\psi(x)$ is false for all independent variables.*

Operation of Sets

Definition. *The* ***union*** *of set A and set B is defined as*

$$A \cup B := \{x \mid x \in A \vee x \in B\}$$

Definition. *The* ***intersection*** *of set A and set B is defined as*

$$A \cap B := \{x \mid x \in A \wedge x \in B\}$$

Definition. *The* ***complement*** *of set A with respect to U ($A \subset U$) is defined as*

$$\complement_U A := \{x \mid x \in U \wedge x \notin A\}$$

Map and Structure

Definition. ***Cartesian Product*** *of set X and set Y is defined as*

$$X \times Y := \{(x, y) \mid x \in A, y \in B\}$$

Definition. ***Graph*** *$\mathcal{G}$ of X with respect to Y is a subset of $X \times Y$ satisfying*

$$|(x, y) \in \mathcal{R}| \wedge |(x, y') \in \mathcal{G}| \Rightarrow y = y'$$

where = means the independent variables in two sides are the same variable.

Definition. ***Map*** *f is a graph $r \subset X \times Y$ together with set X and set Y, denoted by*

$$f : X \to Y \ \text{ or } \ f : x \mapsto y$$

where $x \in X$ and $y \in Y$.

Note. The **value** of map f at variable x is denoted by $f(x)$ from now on.

Definition. ***Domain*** $\mathrm{dom}(f)$ *of a map $f : X \to Y$ is the set X, denoted by*

$$\mathrm{dom}(f) := X$$

Definition. ***Codomain*** $\mathrm{cod}(f)$ *of a map $f : X \to Y$ is the set Y, denoted by*

$$\mathrm{cod}(f) := Y$$

Definition. ***Image*** $\mathrm{img}(f)$ *of a map $f : X \to Y$ is defined as*

$$\mathrm{img}(f) := \{y \in Y \mid (\exists x \in X)\, f(x) = y\}$$

Definition. ***Image*** *of a map $f : X \to Y$ to set $U \subset X$ is defined as*

$$\mathrm{img}(f)[U] \equiv f[U] := \{y \in Y \mid (\exists x \in U)\, f(x) = y\}$$

Note. Symbol $\equiv$ refers to "denoted by".

Note. Previously, there is a concept called *range* of a map. However, *range* has different definitions in different documents. Thus, in order to coincide with other documents, concept *range* is deactivated.

Definition. *The* ***restriction*** *of map* $f : X \to Y$ *to* $\tilde{X} \subset X$ *is defined as*

$$\tilde{f} : \tilde{x} \mapsto y \ \wedge \ \tilde{x} \in \tilde{X}$$

Definition. *The* ***composite*** *of map* $f : X \to Y$ *and map* $g : \tilde{Y} \to Z$ *is defined as*

$$[g \circ f](x) := g(f(x))$$

if $\mathrm{dom}(g) \subset \mathrm{img}(f)$.

Definition. *Two maps* f *and* f' *are* ***equal*** *if*

(i) $\mathrm{dom}(f) = \mathrm{dom}(f')$

(ii) $\mathrm{graph}(f) = \mathrm{graph}(f')$

Definition. *Map* $f : x \mapsto y$ *is* ***injective*** *if*

$$(\forall x, x') :: f(x) = f(x') \Rightarrow x = x'$$

Definition. *Map* $f : x \mapsto y$ *is* ***surjective*** *if*

$$(\forall y \in Y, \exists x \in X) :: f(x) = y$$

Definition. *Map* f *is* ***bijective*** *if it is injective and surjective.*

Note. Symbol $::$ refers to "such that".

Note. A map that is injective, surjective or bijective is also called **injection**, **surjection** or **bijection** respectively.

Definition. ***Constant map*** f *is defined as*

$$(\forall x, x' \in X) :: f(x) = f(x')$$

Definition. ***Identity map*** id_X *of set* X *is defined as*

$$\mathrm{id}_X : x \mapsto x$$

Definition. ***Inverse image*** *of map* $f : X \to Y$ *to set* $\tilde{Y} \subset Y$ *is defined as*

$$\mathrm{img}^{-1}(f)[\tilde{Y}] \equiv f^{-1}[\tilde{Y}] := \{ x \in X \mid f(x) \in \tilde{Y} \}$$

Definition. ***Inverse map*** $f^{-1} : \mathrm{img}(f) \to X$ *of injection* $f : X \to Y$ *is defined as*

$$| f^{-1}(y) := x \,|\, \wedge \,|\, f(x) = y \,|$$

Connotation. ***Mathematical structure*** Σ *on set* X *is a subset of* $\mathcal{S}(X)$ *where* $\mathcal{S}(X)$ *is a set generated by* X.

There is an equivalent definition of mathematical structure on set

Connotation. ***Mathematical structure*** σ *on set* X *is a map* $\sigma : \mathcal{S}(X) \to \{\mathit{true}, \mathit{false}\}$ *where* $\mathcal{S}(X)$ *is a set generated by* X *and* $\{\mathit{true}, \mathit{false}\}$ *is the set of true and false.*

Note. It is easy to verify the equivalence of the above definitions. Set $\{x \mid \sigma(x)\}$ can be the set needed in the first definition; and if there is a chosen subset Σ of $\mathcal{S}(X)$, σ maps every element of the subset to true and the rest to false.

Connotation. ***Morphism*** $m : X \to Y$ *from structured set* (X, σ) *to* (Y, τ) *is a bijection satisfying*

$$\sigma(x) \Leftrightarrow \tau[m(x)]$$

Definition. ***Family of sets*** $\mathcal{F}$ *is a class with all objects being sets.*

Note. A family of sets may be a proper class.

Definition. ***Power set*** $\mathcal{P}$ *of set* X *is defined as*

$$\mathcal{P} := \{ \tilde{X} \mid \tilde{X} \subset X \}$$

Theorem. *Power set is a set.*

Proof. There exists a class $\mathcal{P}$ such that $\mathcal{P} \subset \mathcal{P}$. □

1.1.3 Category Theory

Definition. ***Indexing function*** ι *of family of sets* $\mathcal{F}$ *indexed by non-empty set* I *is a surjection*

$$\iota : i \to X_i$$

where $i \in I, X_i \in \mathcal{F}$, *non-empty refers to* $I \neq \varnothing$.

Note. A family of sets $\mathcal{F}$ is **indexed** (by I) if there defines an indexing function from I to $\mathcal{F}$.

Definition. ***Union*** *of sets in indexed family is defined as*

$$\bigcup_{i \in I} X_i := \{ x \mid (\exists i) :: x \in X_i \}$$

Definition. ***Intersection*** *of sets in indexed family is defined as*

$$\bigcap_{i \in I} X_i := \{ x \mid (\forall i) :: x \in X_i \}$$

Definition. ***Cartesian product*** *of sets in indexed family is defined as*

$$\prod_{i\in I} X_i := \{m : I \to X_i \,|\, (\forall i)\ m(i) \in X_i\}$$

Definition. ***Category*** $C(\mathrm{ob}, \mathrm{mph})$ *is a class* $\mathrm{ob}(C)$ *of objects together with a set* $\mathrm{mph}(X, Y)$ *of morphisms between object X and Y.*

Definition. ***Functor*** *F from category C to category* $\mathcal{D}$ *is a map* $\mathrm{ob}F : \mathrm{ob}C \to \mathrm{ob}\mathcal{D}$ *together with a map* $\mathrm{mph}F : \mathrm{mph}C \to \mathrm{mph}\mathcal{D}$ *satisfying*

(i) $\mathrm{mph}F(f \circ g) = \mathrm{mph}F(f) \circ \mathrm{mph}F(g)$

(ii) $\mathrm{mph}F(\mathrm{id}_X) = \mathrm{id}_{\mathrm{ob}F(X)}$

Definition. ***Functor Transformation*** τ *from functor F to functor G is a set of morphisms*

$$\tau := \{\tau_X \in \mathrm{mph} \mid \tau_X : \mathrm{ob}F(X) \to \mathrm{ob}G(X)\}$$

Definition. *A functor transformation* $\tau : F \to G$ *is* ***natural*** *if*

$$(\forall X, Y \in \mathrm{ob}, f \in \mathrm{mph}) :: \tau_X \circ \mathrm{mph}F(f) = \mathrm{mph}G(f) \circ \tau_Y$$

1.2 Algebraic Structure

This section illustrates mathematical structure of algebra.

1.2.1 Group Structure

Definition. ***Binary operation*** *on set X is a map* $f : X \times X \to X$.

Note. In binary operation, notion $f(x, y)$ is usually replaced by xfy.

Note. Binary operation is a mathematical structure.

Definition. ***Product*** $\cdot$ *on a set X is a binary operation on set X.*

Note. Usually, product $x \cdot y$ can be abbreviated to xy.

Definition. ***Semi-group*** $\hat{G}$ *is a non-empty set together with a product* $\cdot$ *satisfying associative condition*

$$(\forall g, h, k \in \hat{G}) :: (gh)k = g(hk)$$

Definition. ***Monoid*** $\hat{M}$ *is a semi-group satisfying*

$$(\exists e \in \hat{M}) :: ge = eg = g$$

Note. Element e is called **identity element** of monoid $\hat{M}$.

Definition. ***Group*** *G is a monoid satisfying*

$$(\forall g \in G, \exists h \in G) :: gh = hg = e$$

Note. Element h is called **inverse element** of element g, denoted by g^{-1}.

Theorem. *The identity element of a monoid and inverse element of an element in a group are unique.*

Proof. There are the following argument

(i) If e' and e both are the identity element, then

$$e = ee' = e'$$

(ii) If h and k both are the inverse element of g then

$$h = he = hgk = ek = k$$

which ends the proof. □

Definition. *Group G is* ***abelian*** *if*

$$(\forall g, h \in G) :: gh = hg$$

Note. Abelian group is also called **commutative** group.

1.2.2 Ring Structure

Definition. ***Ring*** $(R, +, \cdot)$ *is a non-empty set R together with binary operation* ***addition*** $+$ *and* ***multiplication*** $\cdot$ *satisfying*

(i) Addition: $(R, +)$ *is an abelian group*

(ii) Multiplication: $(R, \cdot)$ *is a semi-group.*

(iii) Distribution: $(\forall r, s, t \in R) :: r \cdot (s + t) = r \cdot s + r \cdot t$

Note. The last condition indicates the compatibility of addition and multiplication.

Definition. *Ring* $(R, +, \cdot)$ *is* ***commutative*** *if*

$$(\forall r, s \in R) :: r \cdot s = s \cdot r$$

Definition. *Ring* $(R, +, \cdot)$ *is a* ***ring with identity*** *if* $(R, \cdot)$ *is a monoid.*

Definition. *An element r in a ring* $(R, +, \cdot)$ *with identity is a* ***unit*** *if it has a multiplicative inverse.*

Definition. *A ring* $(R, +, \cdot)$ *with identity is a* ***division ring*** *if every element except the identity element of addition is a unit.*

Definition. ***Field*** *is a commutative division ring.*

1.3 Number Theory

This section illustrates the construction of numbers.

1.3.1 Relation Structure

Definition. ***Relation*** *$\mathcal{R}$ of set X is a subset of $X \times X$*

Note. Usually, that element x and element y has relation $\mathcal{R}$, i.e. $(x, y) \in \mathcal{R}$, is denoted as $x\mathcal{R}y$.

Note. Relation is a mathematical structure on set.

Definition. ***Equivalence relation*** *$\sim$ of set X is a relation satisfying*

(a) Reflexivity: $(\forall x{\in}X) :: x \sim x$

(b) Symmetry: $x \sim y \Rightarrow y \sim x$

(c) Transitivity: $|x \sim y| \wedge |y \sim z| \Rightarrow x \sim z$

Definition. ***Equivalence class*** *$E|_x$ of a set X with equivalence relation $\sim$ determined by x is defined by*

$$E|_x := \{y{\in}X \mid y \sim x\}$$

Note. According to Semiotics, equivalence class should be phrased as *equivalence set* since it can be easily proven that the equivalence class is a set. However, due to some historical reasons, concept *equivalence set* was not invented. Thus, to coincide with other documents, equivalence class is used in this document.

Note. Two sets X and Y are disjoint if $X \cap Y = \varnothing$.

Theorem. *Two equivalence class $E|_x$ and $E'|_{x'}$ can be either equal or disjoint.*

Proof. There are the following arguments

(i) If $E|_x \cap E'|_{x'} = \varnothing$, then they are disjoint

(ii) If $E|_x \cap E'|_{x'} \neq \varnothing$, suppose $y \in E|_x \cap E'|_{x'} = \varnothing$ and there is

$$|\, y \sim x \,| \wedge |\, y \sim x' \,| \Rightarrow |\, x \sim y \,| \wedge |\, y \sim x' \,| \Rightarrow x \sim x'$$

Thus $(\forall w \in E|_x)\, w \in E'|_{x'}$, which means $E|_x \subset E'|_{x'}$ Reverse $E|_x$ and $E'|_{x'}$ to get $E'|_{x'} \subset E|_x$ Thus $E|_x$ and $E'|_{x'}$ are equal

Thus, the theorem is proved. □

Definition. ***Partition*** *$\mathcal{D}$ of set X is a family of sets with all objects being non-empty disjoint subsets of X.*

Theorem. *A partition is a set.*

Proof. There exists a class $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{D}$. □

Note. If define every element of a partition as an equivalence class, there will be a bijection between the partition and the equivalence class.

Note. Quotient space $X/\!\sim$ is a partition formed by $\sim$.

Definition. ***Order relation*** *$\prec$ of set X is a relation satisfying*

(a) Comparability:

$$|x \prec y| \wedge |y \prec x| = \mathit{flase}$$
$$|x \prec y| \vee |y \prec x| = \mathit{true}$$

(b) Non-reflexivity: $x \prec x = \mathit{false}$

(c) Transitivity: $(x \prec y) \wedge (y \prec z) \Rightarrow x \prec z$

Definition. ***Partial order relation*** *$\preccurlyeq$ of set X is a relation satisfying*

(a) Reflexivity: $x \preccurlyeq x$

(b) Anti-symmetry: $|x \preccurlyeq y| \wedge |y \preccurlyeq x| \Rightarrow x = y$

(c) Transitivity: $|x \preccurlyeq y| \wedge |y \preccurlyeq z| \Rightarrow x \preccurlyeq z$

Definition. ***Total order relation*** *$\preccurlyeq$ of set X is a relation satisfying*

(a) $\preccurlyeq$ is a partial order relation

(b) Comparability:

$$|x \preccurlyeq y| \wedge |y \preccurlyeq x| = \mathit{flase}$$
$$|x \preccurlyeq y| \vee |y \preccurlyeq x| = \mathit{true}$$

Note. The order relation and partial order relation can be disparate. There are no subset interconnections in between. However, partial order relation and total order relation have a clear including interconnection, i.e. a total order relation must be a partial order relation.

Note. A set with (partial or total) order relation is called **(partially or totally) ordered set**.

Definition. ***Open interval*** *(a, b) in an ordered set $(X, \prec)$ is defined as*

$$(a, b) := \{\, x{\in}X \mid a \prec x \prec b \,\}$$

Note. If $(a, b) = \varnothing$, then a is the **immediate predecessor** of b and b is the **immediate successor** of a.

Definition. ***Alphabetical order relation*** *$\prec$ of Cartesian product $X \times Y$ of two ordered set $(X, \prec|_X)$ and $(Y, \prec|_Y)$ is defined as*

$$(x_1, y_1) \prec (x_2, y_2) \Leftrightarrow (x_1 \prec x_2) \vee [\,|x_1 = x_2| \wedge |y_1 \prec y_2|\,]$$

Note. Alphabetical partial or total order relation of Cartesian product can be defined through similar ways.

Note. Element b is the **largest element** of partially ordered set $(X, \preccurlyeq)$ if $(\forall x \in X) :: x \preccurlyeq b$. Similarly, element a is the **smallest element** of partially ordered set $(X, \preccurlyeq)$ if $(\forall x \in X) :: a \preccurlyeq x$.

Note. Subset $\tilde{X}$ of partially ordered set $(X, \preccurlyeq)$ is **bounded above** if $(\exists b \in X, \forall \tilde{x} \in \tilde{X}) :: \tilde{x} \preccurlyeq b$ and b is the **upper bound**. Similarly, subset $\tilde{X}$ of partially ordered set $(X, \preccurlyeq)$ is **bounded below** if $(\exists a \in X, \forall \tilde{x} \in \tilde{X}) :: a \preccurlyeq \tilde{x}$ and a is the **lower bound**.

Definition. ***Supremum*** sup *is the smallest element of upper bound.* ***Infimum*** inf *is the largest element of the lower bound.*

Note. A set X has **supremum property** (*resp* **infimum propert**) if for any non-empty subset of X which is bounded above (*resp* below) has an supremum (*resp* infimum).

1.3.2 Real Number Field

Definition. ***Real number set*** $\mathbb{R}$ *is a totally ordered field satisfying*

(a) $(\forall x, y, z \in \mathbb{R}) :: x \preccurlyeq y \Rightarrow x + z \preccurlyeq y + z$

(b) $(\forall x, y \in \mathbb{R}) :: |x \preccurlyeq y| \wedge |z > 0| \Rightarrow x \cdot z \preccurlyeq y \cdot z$

(c) Order relation $\preccurlyeq$ *has supremum property*

Definition. *A subset A of* $\mathbb{R}$ *is* ***inductive*** *if*

(a) $1 \in A$

(b) $(\forall x \in A) :: x + 1 \in A$

Definition. ***Positive integer set*** $\mathbb{Z}_+$ *is defined as*

$$\mathbb{Z}_+ := \bigcap_{i \in \mathcal{A}} A_i$$

where $\mathcal{A}$ *is the set of all inductive subsets in* $\mathbb{R}$ *and* A_i *is inductive subset of* $\mathbb{R}$

Note. Section of positive integer $\mathbb{Z}_n$ (n-**tuple**) is

$$\mathbb{Z}_n := \{ z \in \mathbb{Z}_+ \mid 1 \preccurlyeq z \preccurlyeq n \}$$

Definition. ***Integer set*** $\mathbb{Z}$ *is defined as*

$$\mathbb{Z} := \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_-$$

where

$$\mathbb{Z}_- := \{ -x \mid x \in \mathbb{Z}_+ \}$$

Definition. ***Rational number*** $\mathbb{Q}$ *is defined as*

$$\mathbb{Q} := \{ x \cdot y^{-1} \mid |x, y \in \mathbb{Z}| \wedge |y \neq 0| \}$$

Definition. ***Open interval*** (x, y) $(-\infty, y)$, $(x, +\infty)$ *is defined as*

$$(x, y) := \{ z \mid x < z < y \}$$
$$(-\infty, y) := \{ z \mid z < y \}$$
$$(x, +\infty) := \{ z \mid x < z \}$$

Definition. ***Closed interval*** $[x, y]$, $(-\infty, y]$, $[x, +\infty)$ *is defined as*

$$[x, y] := \{ z \mid x \preccurlyeq z \preccurlyeq y \}$$
$$(-\infty, y] := \{ z \mid z \preccurlyeq y \}$$
$$[x, +\infty) := \{ z \mid x \preccurlyeq z \}$$

Note. There are also definitions for half open and half closed interval. But they are not adopted in this document.

Note. Sometimes the real number set is written as $(-\infty, +\infty)$.

1.3.3 Finiteness and Countability

Definition. *A set X is* ***finite*** *if it is empty or*

$$(\exists f : X \to \mathbb{Z}_n) :: f \text{ is bijective}$$

Definition. *A set is* ***infinite*** *if it is not finite.*

Definition. ***Cardinality*** *of an empty set or a non-empty finite set X is defined as*

(i) $\operatorname{card}(\varnothing) = 0$

(ii) $\operatorname{card}(X) = n$ *if* $\exists$ *bijection* $f : X \to \mathbb{Z}_n$

Theorem. $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ *are infinite.*

Theorem. *The cardinality of a set is unique.*

Theorem. *The subset of a finite set is finite.*

Corollary. *For finite set X and Y, there is*

$$X \subset Y \Leftrightarrow \operatorname{card}(X) < \operatorname{card}(Y)$$

Note. A finite operation of an indexed family of sets means the family of sets is indexed by a finite set.

Theorem. *A finite union (Cartesian product) of finite sets are finite.*

Definition. *A set is* ***countably infinite*** *if*

$$(\exists f : X \to \mathbb{Z}_+) :: f \text{ is bijective}$$

Definition. *A set is* ***countable*** *if it is finite or countably infinite.*

Definition. *A set is* ***uncountable*** *if it is not countable.*

Theorem. *The subset of a countable set is countable.*

Corollary. $\mathbb{Z}_+, \mathbb{Q}$ *are countably infinite.*

Note. A countable operation of an indexed family of sets means the family of sets is indexed by a countable set.

Theorem. *A countable union of countable set is countable. A finite product of countable set is countable.*

Note. Notation X^n and X^ω with X being a set means

$$X^n \equiv \prod_{i \in \mathbb{Z}_n} X_i, \quad X^\omega \equiv \prod_{i \in \mathbb{Z}_+} X_i$$

Corollary. *There does not exist an injective $f : \mathcal{P}(X) \to X$ or a surjective $f : X \to \mathcal{P}(X)$ where X is a set and $\mathcal{P}(X)$ is the power set.*

2 General Topology

This chapter constructs the topological structure on sets. Default space refers to topological space.

2.1 Topological Space

Definition. ***Topological space*** *$(X, \mathcal{T})$ is a non-empty set X together with a subset $\mathcal{T}$ of its power set satisfying*

(i) Minimum collection

$$\varnothing, X \in \mathcal{T}$$

(ii) Union Property indexed by any indexing set J

$$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$$

(iii) Intersection Property indexed by $\mathbb{Z}_n$

$$\bigcap_{i \in \mathbb{Z}_n} U_i \in \mathcal{T}$$

where J is a set of indexes.

Note. Family of sets $\mathcal{T}$ of topological space $(X, \mathcal{T})$ is the **topology** of set X.

Note. The indexing set for union property and intersection property are different. The indexing set for union can be an arbitrary indexing set, but that for intersection can only be a finite and countable set.

Note. Topology is a mathematical structure.

Note. A subset $\tilde{X}$ of X with $\mathcal{T}$ is **open** if $\tilde{X} \in \mathcal{T}$

Note. A subset $\tilde{X}$ of X with $\mathcal{T}$ is **closed** if $\complement_X \tilde{X} \in \mathcal{T}$

Note. If $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are topological space, $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ or $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$ if $\mathcal{T}_1 \subset \mathcal{T}_2$

Definition. ***Basis*** *of topology $\mathcal{B}$ on set X is a subset of power set of X satisfying*

(i) Covering condition

$$(\forall x \in X, \exists B \in \mathcal{B}) :: x \in B$$

(ii) Intersection property

$$|B_i \in \mathcal{B}| \wedge |x \in \bigcap_{i \in \mathbb{Z}_n} B_i| \Rightarrow (\exists B \in \mathcal{B}) :: x \in B \subset \bigcap_{i \in \mathbb{Z}_n} B_i$$

Definition. ***Topology $\mathcal{T}$ generated by basis*** *$\mathcal{B}$ on set X is defined as*

$$\mathcal{T} := \{\, U \subset X \mid (\forall x \in U, \exists B \in \mathcal{B}) :: x \in B \subset U \,\}$$

Note. This means that the topology is generated by all possible union of basis element. This is valid since the basis will satisfy the requirement of intersection.

Note. The topology generated by basis is uniquely determined by the basis. However, there may exist many bases that generate the same topology generated by basis.

Theorem. *Topology generated by basis is a topology.*

Proof. There are the following arguments

(i) For $\varnothing$ the condition is always true since no elements are contained in $\varnothing$. For X, the first condition of definition of basis insures $X \in \mathcal{T}$

(ii) Consider the union

$$U = \bigcup_{i \in J} U_i, U_i \in \mathcal{T}$$

By definition, $(\forall x \in U, \exists i \in J) :: x \in U_i$
Also, $U_i \in \mathcal{T} \Rightarrow (\exists B \in \mathcal{B}) :: x \in B \subset U_i$
Thus, $(\forall x \in U, \exists B \in \mathcal{B}) :: x \in B \subset U$

(iii) Consider the intersection

$$U = \bigcap_{i\in\mathbb{Z}_n} U_i, U_i \in \mathcal{T} \text{ and } B_I = \bigcap_{i\in\mathbb{Z}_n} B_i, B_i \in \mathcal{B} \subset U_i$$

First, $(\forall x \in U, \forall i \in \mathbb{Z}_n, \exists B_i \in \mathcal{B}) :: x \in B_i \subset U_i$
As a consequence, $x \in B_I \subset U$
By definition, $(\exists B \in \mathcal{B}) :: x \in B \subset B_I$
Thus, $(\forall x \in U, \exists B \in \mathcal{B}) :: x \in B \subset U$

The arguments show that the topology generated by basis satisfies the condition of a topology, and thus end the proof. □

Note. A subset $\mathcal{S}$ of power set of X **covers** X if

$$(\forall x, \exists S \in \mathcal{S}) :: x \in S)$$

Definition. ***Subbasis*** *$\mathcal{E}$ for topology on set X is a subset of power set of X that covers X.*

Note. There are several definitions for subbasis for topology and not all of them are equivalent.

Definition. ***Basis $\mathcal{B}$ associated with subbasis*** *$\mathcal{E}$ on set X is defined as*

$$\mathcal{B} := \{ \bigcap_{i\in\mathbb{Z}_n} E_i \mid E_i \in \mathcal{E}, n \in \mathbb{Z}_+ \}$$

Note. This means that the basis is generated by all intersection of subbasis.

Note. The validity of the above definition can be checked by verifying the properties of basis. The first condition is satisfied since the subbasis covers the set; and the second condition is satisfied as the basis element is the intersection of subbasis elements.

Note. Topology generated by subbasis refers to topology generated by basis associated with the subbasis.

Corollary. *Topology generated by subbasis is a topology.*

Proof. Apparently. □

Definition. ***Local base*** *$\mathcal{B}(x)$ is a subset of $\mathcal{N}(x)$ satisfying*

$$(\forall N \in \mathcal{N}(x), \exists B \in \mathcal{B}(x)) :: B \subset N$$

Definition. ***Natural Projection*** *on Cartesian product of indexed family of sets is defined as*

$$p_i : x \mapsto m_x(i)$$

where x is the element and m_x is the corresponding map in the definition of Cartesian product.

Definition. ***Product topology*** *$\mathcal{K}$ of Cartesian product of indexed family of topological space*

$$\prod_{i\in I}(X_i, \mathcal{T}_i)$$

is defined as a topology generated by a subbasis

$$\mathcal{K} := \text{span}\{ \text{img}^{-1}(p_i)[O_i] \mid O_i \in \mathcal{T}_i, i \in I \}$$

Note. Box topology is another candidate for the topology of product set. However, it fails to retain some good properties such as the continuity (defined later) of projection map so that it is eliminated.

Note. Element in a topological space is also called point.

Definition. ***Induced topology*** *$\mathcal{S}$ of a subset $\tilde{X}$ of a topological space $(X, \mathcal{T})$ is defined as*

$$\mathcal{S} := \{ O \cap \tilde{X} \mid O \in \mathcal{T} \}$$

Note. $(\tilde{X}, \mathcal{S})$ forms a **topological subspace** of $(X, \mathcal{T})$.

Definition. ***n-dimensional Euclidean space*** *$\mathbb{R}^n$ is defined as*

$$\mathbb{R}^n := \prod_{i\in\mathbb{Z}_n} \mathbb{R}_i$$

Note. Usually, the product symbol $\cdot$ in $\mathbb{R}$ is omitted.

Definition. ***Distance*** *of two point x and y in n-dimensional Euclidean space is defined as*

$$\delta(x, y) := \sqrt{\sum_{i\in\mathbb{Z}_n}(x^i - y^i)(x_i - y_i)}$$

where x^i means $m_x(i)$ as is in the definition of Cartesian product of indexed family of sets.

Note. Hereby, $x = \sqrt{y}$ refers to $x \cdot x = y$.

Note. In the above definition, a notation is used

$$\sum_{i\in\mathbb{Z}_n} \psi_i := \psi_n + \sum_{i\in\mathbb{Z}_{n-1}} \psi_i \,, \sum_{i\in\mathbb{Z}_1} \psi_i = \psi_1$$

Definition. ***ε-open ball*** *of point p in n-dimensional Euclidean space is defined as*

$$B(p, \varepsilon) := \{ q \mid \delta(p, q) < \varepsilon \}$$

where $\varepsilon \in \mathbb{R}$.

Definition. ***Usual topology*** *of n-dimensional Euclidean space is the topology generated by subbasis being the set of all open balls.*

Definition. ***Neighbourhood*** *N at point x of topological space $(X, \mathcal{T})$ is a subset of X satisfying*

$$(\exists O \in \mathcal{T}) :: x \in O \subset N$$

Note. **Open neighbourhood** N is a neighbourhood that is open.

Definition. ***Neighbourhood system*** *$\mathcal{N}(p)$ of point p of a topological space is the set of all neighbourhood of the point.*

Theorem. *Subset $U \subset X$ is open $\Leftrightarrow (\forall x \in U) :: U \in \mathcal{N}(x)$*

Proof. There are the following arguments

(i) If $U \in \mathcal{T}$, then $(\forall x \in U, \exists U \in \mathcal{T}) :: x \in U \subset U$

(ii) If $(\forall x \in U) :: U \in \mathcal{N}(x)$, then

$$U = \bigcup_{x \in U} U_x \in \mathcal{T}$$

where $U_x \in \mathcal{T}$ satisfies $x \in U_x \subset U$

These arguments end the proof. □

Definition. *Map $f : X \to X'$ of topological space $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ is **continuous** if*

$$(\forall U' \in \mathcal{T}') :: \mathrm{img}^{-1}(f)[U'] \in \mathcal{T}$$

Note. Map f is C^0 means it is continuous.

Definition. *Map $f : X \to X'$ of topological space $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ is **continuous at point** $x \in X$ if*

$$(\forall f(x) \in U' \wedge U' \in \mathcal{T}', \exists U \in \mathcal{T}) :: x \in U \wedge f[U] \subset U'$$

The next theorem needs to invoke term "if and only if" which means $\Leftrightarrow$. Proving $\psi \Leftrightarrow \varphi$ is usually divided into proving $\psi \Rightarrow \varphi$ and $\varphi \Rightarrow \psi$. The former is called **Sufficiency** and the latter is called **Necessity**.

Theorem. *Map f is continuous if and only if it is continuous at every point in domain.*

Proof. There are the following arguments

(i) Sufficiency. Provided f is continuous. Then $\forall x \in \mathrm{dom}(f)$, suppose $f(x) \in O' \wedge O' \in \mathcal{T}'$, there is

$$(\exists \mathrm{img}^{-1}(f)[O'] \in \mathcal{T}) :: \\ x \in \mathrm{img}^{-1}(f)[O'] \wedge \mathrm{img}(f)[\mathrm{img}^{-1}(f)[O']] \subset O'$$

Thus, f is continuous at every point in domain.

(ii) Necessity. Provided f is continuous at every point in domain. Suppose $O' \in \mathcal{T}'$
If $O = \varnothing$, then $\mathrm{img}^{-1}(f)[O] = \varnothing \in \mathcal{T}$
If $O \neq \varnothing$, then

$$(\forall x \in \mathrm{img}^{-1}(f)[O']) :: f(x) \in O' \wedge O' \in \mathcal{T}'$$

thus $(\exists U \in \mathcal{T}) :: x \in U \subset \mathrm{img}^{-1}(f)[U]$
which means

$$(\forall x \in \mathrm{img}^{-1}(f)[O']) :: \mathrm{img}^{-1}(f)[O'] \in \mathcal{N}(x)$$

therefore $\mathrm{img}^{-1}(f)[O'] \in \mathcal{T}$

These arguments end the proof □

Corollary. *The composite $f \circ g$ of two continuous map f and g is continuous.*

Corollary. *Restriction of a continuous map is continuous.*

Definition. ***Homeomorphism*** *from topological space $(X, \mathcal{T}_x)$ to $(Y, \mathcal{T}_y)$ is a map $f : X \to Y$ satisfying*

(i) Map f is bijective

(ii) Map f and f^{-1} are continuous

Note. A homeomorphism is also called homeomorphic map.

Definition. *Topological space $(X, \mathcal{T}_x)$ to $(Y, \mathcal{T}_y)$ are **homeomorphic** to each other if there exists a homeomorphism in between.*

Theorem. *The inverse of homeomorphism is a homeomorphism.*

Theorem. *The composite of homeomorphisms is a homeomorphism.*

2.2 Topological Properties

Connotation. ***Topological Property*** *is a predicate on topological space satisfying*

$$\psi_X \Leftrightarrow \psi_Y$$

where X, Y are two homeomorphic topological space.

Note. Topological property is also called **topological invariant**.

2.2.1 Separation

Definition. *The **sequence** $\{x_i\}$ of set X is a map $\{x_i\} : \mathbb{Z}_+ \to X$ where $x \in X$ and $i \in \mathbb{Z}_+$.*

Definition. *The **limit** of a sequence $\{x_i\}$ is defined as*

$$\lim_{i\to\infty} x_i := x \Leftrightarrow (\forall U \in \mathcal{N}(x), \exists n_c \in \mathbb{Z}_+) :: i > n_c \Rightarrow x_i \in U$$

Note. That the limit of sequence $\{x_i\}$ is x is also phrased that $\{x_i\}$ converges to x.

Note. The sequence is **convergent** if it has a limit.

Definition. *Point x is the **limit point** of subset $\tilde{X}$ of space $(X, \mathcal{T})$ if*

$$(\forall N \in \mathcal{N}(x)) :: N \cap \tilde{X} \cap \complement_{\tilde{X}}\{x\} \neq \varnothing$$

Definition. *Space $(X, \mathcal{T})$ is **Fréchet** or T_1 if*

$$(\forall x \neq y \in X, \exists U \in \mathcal{N}(x)) :: y \notin U$$

Definition. *Space $(X, \mathcal{T})$ is **Hausdorff** or T_2 if*

$$(\forall x \neq y \in X, \exists U_x \in \mathcal{N}(x), U_y \in \mathcal{N}(y)) :: U_x \cap U_y = \varnothing$$

Theorem. *If $(X, \mathcal{T})$ is Hausdorff, then a sequence of X converges to no more than one point in X.*

2.2.2 Countability

Definition. *Subset $\tilde{X}$ of space $(X, \mathcal{T})$ is **dense** in X if* $\mathrm{cls}(\tilde{X}) = X$.

Definition. *A topological space is **first-countable** if every point of it has a countable local base.*

Definition. *A topological space is **second-countable** if it has a countable base.*

Theorem. *The subspace of first-countable (second-countable) space is first-countable (second-countable).*

Theorem. *The finite product of first-countable (second-countable) space is first-countable (second-countable).*

2.2.3 Connectedness

Definition. ***Closure*** $\mathrm{cls}(S)$ *of a subset S of topological space $(X, \mathcal{T})$ is defined as*

$$\mathrm{cls}(S) := \bigcap_{\alpha\in I} \tilde{X}_\alpha \mid S \subset \tilde{X}_\alpha \wedge \tilde{X}_\alpha \in \mathcal{T}$$

Definition. ***Interior*** $\mathrm{int}(S)$ *of a subset S of topological space $(X, \mathcal{T})$ is defined as*

$$\mathrm{int}(S) := \bigcup_{\alpha\in I} \tilde{X}_\alpha \mid S \supset \tilde{X}_\alpha \in \mathcal{T}$$

Definition. ***Boundary*** ∂S *of a subset S of topological space $(X, \mathcal{T})$ is defined as*

$$\partial S := \mathrm{cls}(S) \cap \complement_X \mathrm{int}(S)$$

Theorem. *Set X and $\varnothing$ are both closed and open in topological space $(X, \mathcal{T})$.*

Definition. *Topological Space $(X, \mathcal{T})$ is **connected** if X and $\varnothing$ are the only set that is both open and closed.*

Theorem. *Topological Space $(X, \mathcal{T})$ is **connected** if*

$$(\nexists\{U, V\} \subset \mathcal{T}) :: \{U, V\} \textit{ is a partition}$$

Note. Symbol $\nexists$ refers to $\neg\exists$.

Note. This theorem explains the reason for the name connectedness.

Theorem. *The union of connected subspaces of a topological space is connected if their intersection is non-empty.*

Theorem. *The image of a connected space under a continuous map is connected.*

Proof. There are the following arguments:
For continuous map $f : X \to Y$ with $(X, \mathcal{T})$ connected, assume that $\mathrm{img}[f]$ is not connected, i.e.

$$(\nexists\{U, V\} \subset \mathcal{T}_{\mathrm{img}[\{]}) :: U \cup V = \mathrm{img}(X)$$

there will be

$$f^{-1}[U] \cap f^{-1}[V] \text{ is a partition } \wedge f^{-1}[U], f^{-1}[V] \in \mathcal{T}$$

which indicates X is not connected and contradicts the previous condition. Therefore, $\mathrm{img}(X)$ is connected. □

Corollary. *Connectedness is a topological property.*

Proof. Apparently. □

2.2.4 Compactness

Definition. *Family of sets C is an **open covering** of space $(X, \mathcal{T})$ if it covers X and $C \subset \mathcal{T}$.*

Definition. *Topological space $(X, \mathcal{T})$ is **compact** if every open covering of X has a finite subset that also covers X.*

Theorem. *Every closed subspace of a compact space is compact.*

Theorem. *Every compact subspace of a Hausdorff space is closed.*

Theorem. *The image of a compact space under a continuous map is compact.*

Proof. For continuous map $f : X \to Y$, compact space $(X, \mathcal{T})$, $n \in \mathbb{Z}_+$ and an open covering C of $\text{img}(X)$

$$(\exists \widetilde{f^{-1}[C]} \subset f^{-1}[C]) :: |\text{card}(\widetilde{f^{-1}[C]}) = n| \wedge |\widetilde{f^{-1}[C]} \text{ covers } X|$$

Thus, $f[\widetilde{f^{-1}[C]}]$ is a finite open covering of $\text{img}(X)$ since

$$(\forall y \in \text{img}(X), \exists x) :: f(x) = y$$

which ends the proof. □

Note. Symbol $f[\mathcal{F}]$ of a family of sets refers to

$$f[\mathcal{F}] := \{f[F] \mid F \in \mathcal{F}\}$$

Theorem. *Compactness is a topological property.*

Proof. Apparently. □

Theorem. *For bijective continuous function $f : X \to Y$, if X is compact and Y is Hausdorff, then f is a homeomorphism.*

Theorem. *The product of finite compact spaces is compact.*

3 Measure Theory

This chapter cares about general theory of measure structure, which enables the construction of integration on sets. Default space is measure space and default function is set function.

3.1 Measure on Measurable Sets

This section concerns the measurable sets and the measure on them.

3.1.1 Ring and Algebra of Sets

Definition. ***Ring of sets*** *$\mathcal{R}(X)$ is a subset of power set $\mathcal{P}(X)$ of non-empty X satisfying*

$$(\forall E_1, E_2 \in \mathcal{R}(X)) :: |E_1 \cup E_2 \in \mathcal{R}(X)| \wedge |E_1 \cap \complement_X E_2 \in \mathcal{R}(X)|$$

Definition. ***Algebra of sets*** *$\mathcal{A}(X)$ is a ring of set X satisfying $X \in \mathcal{A}(X)$.*

Note. Ring and algebra of sets are not ring or algebraic structure.

Note. Ring and algebra of sets are mathematical structures.

Definition. *A family of sets $\mathcal{F}$ is **closed under binary operation** f if*

$$(\forall U_1, U_2 \in \mathcal{F}) :: U_1 f U_2 \in \mathcal{F}$$

Theorem. *Ring (*algebra*) is closed under intersection.*

Theorem. *Empty set is in ring or algebra of sets.*

Theorem. *For a subset $\mathcal{E}$ of power set $\mathcal{P}(X)$, there is unique ring (*resp *algebra) $\mathcal{R}(\mathcal{E})$ satisfying*

(i) Inclusion $\mathcal{E} \subset \mathcal{R}(X)$
(ii) Minimum $(\forall \mathcal{R}(X) \supset \mathcal{E}) :: \mathcal{R}(\mathcal{E}) \subset \mathcal{R}(X)$

Note. Unique means for every $\mathcal{R}(\mathcal{E})$ and $\mathcal{R}'(\mathcal{E})$, there is $\mathcal{R}(\mathcal{E}) = \mathcal{R}'(\mathcal{E})$.

Definition. *Family of sets $\mathcal{R}(\mathcal{E})$ is the **σ-ring** (*resp ***σ-algebra***) generated by $\mathcal{E}$*

Definition. ***ring*** *(*resp ***algebra****) $\mathcal{R}_\sigma(X)$ is a ring (*resp *algebra) satisfying*

$$\bigcup_{i \in \mathbb{Z}_+} E_i \in \mathcal{R}_\sigma(X) \;\; where \;\; E_i \in X$$

Theorem. *σ-ring (*resp *σ-algebra) is closed under countable intersection.*

Note. Countable intersection refers to intersection of family of sets indexed by $\mathbb{Z}_+$.

Theorem. *For a subset $\mathcal{E}$ of power set $\mathcal{P}(X)$, there is unique σ-ring (*resp *σ-algebra) $\mathcal{R}_\sigma(\mathcal{E})$ satisfying*

(i) Inclusion $\mathcal{E} \subset \mathcal{R}_\sigma(X)$
(ii) Minimum $(\forall \mathcal{R}_\sigma(X) \supset \mathcal{E}) :: \mathcal{R}_\sigma(\mathcal{E}) \subset \mathcal{R}_\sigma(X)$

Definition. *Family of sets $\mathcal{R}_\sigma(\mathcal{E})$ is the **σ-ring** (*resp ***σ-algebra***) generated by $\mathcal{E}$*

3.1.2 Measure on Rings of Sets

Definition. *Sequence $\{x_i\}$ of $\mathbb{R}$ **tends to infinity** if*

$$(\forall \varepsilon > 0, \exists n_c \in \mathbb{Z}_+) :: i > n_c \Rightarrow x_i > \varepsilon$$

Definition. ***Numerical infinity*** *∞ is the limit of a sequence $\{x_i\}$ of $\mathbb{R}$ which tends to infinity, i.e.*

$$\lim_{i\to\infty} x_i = \infty$$

Definition. ***Minus numerical infinity*** *$-\infty$ is the limit of sequence $\{-x_i\}$ with $\{x_i\}$ tending to infinity.*

Definition. ***Extended real number line*** *$\hat{\mathbb{R}}$ is defined as*

$$\hat{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$$

Definition. ***Series*** *is the addition of sequence $\{x_i\}$ of $\mathbb{R}$ defined as*

$$\sum_{i\in\mathbb{Z}_+} x_i := \lim_{n\to\infty} \sum_{i\in\mathbb{Z}_n} x_i$$

Definition. ***Set function*** *of family of sets $\mathcal{F}$ is a map $\mu : \mathcal{F} \to \mathbb{R}$.*

Definition. ***Measure on ring*** *$\mathcal{R}(X)$ is a set function of $\mathcal{R}(X)$ satisfying*

(i) Non-negative $(\forall E \in \mathcal{R}(X)) :: \mu(E) \geqslant 0$

(ii) Countable Additive

$$\mu(\bigcup_{i\in\mathbb{Z}_+} E_i) = \sum_{i\in\mathbb{Z}_+} \mu(E_i)$$

with sequence $\{E_i\}$ satisfying

$$(\forall i, j \in \mathbb{Z}_+) :: E_i \cap E_j = \varnothing, \bigcup_{i\in\mathbb{Z}_+} E_i \in \mathcal{R}(X)$$

Theorem. $\mu(\varnothing) = 0$.

Definition. ***Interval box*** *of $\mathbb{R}$ is a set*

$$\mathcal{I} := \{(x, y] \mid x, y \in \mathbb{R}\}$$

Note. In the above definition, notation $(a, b]$ refers to

$$(a, b] := \{x \in \mathbb{R} \mid a < x \leqslant b\}$$

Definition. ***Usual Measure*** *m on ring $\mathcal{R}(\mathcal{I})$ is defined as*

$$m(E) := \sum_{i\in\mathbb{Z}_n} m(I_i), \quad m(I) := y - x$$

where $I = (x, y] \in \mathcal{I}$ and $\{I_i\}$ is a partition of $E \in \mathcal{R}(\mathcal{I})$ with $I_i \in \mathcal{I}$.

Note. Value y is **irrelevant** to x if map $f : x \mapsto y$ is a constant map.

Theorem. *Value $m(E)$ is irrelevant to the partition of E.*

Proof. For different partition I_i, J_i of set $E \in \mathcal{R}(\mathcal{I})$, there is $G_{ij} \equiv I_i \cap J_j \in \mathcal{I}$. Thus

$$\sum_{i\in\mathbb{Z}_n} m(I_i) = \sum_{i\in\mathbb{Z}_n} \left(\sum_{j\in\mathbb{Z}_n} m(G_{ij}) \right)$$
$$= \sum_{j\in\mathbb{Z}_n} \left(\sum_{i\in\mathbb{Z}_n} m(G_{ij}) \right) = \sum_{j\in\mathbb{Z}_n} m(J_j)$$

which ends the proof. □

Definition. ***Extended set function*** *of family of sets $\mathcal{F}$ is a map $\mu : \mathcal{F} \to \hat{\mathbb{R}}$.*

Definition. ***σ-covering*** *of ring $\mathcal{R}(X)$ is defined as*

$$C_\sigma(\mathcal{R}) := \{E \subset X \mid (\exists\{E_i\}) :: E \subset \bigcup_{i\in\mathbb{Z}_+} E_i\}$$

Lemma. *σ-covering is a σ-ring.*

Definition. ***Outer measure*** *$\hat{\mu}$* ***generated by*** *μ is an extended set function on σ-covering $C_\sigma(\mathcal{R})$ defined as*

$$\hat{\mu}(E) := \inf\{\sum_{i\in\mathbb{Z}_+} \mu(E_i) \mid | E \in \mathcal{R} | \wedge | E \subset \bigcup_{i\in\mathbb{Z}_+} E_i\}$$

Theorem. *The restriction to $\mathcal{R}(X)$ of $\hat{\mu}$ satisfying $\tilde{\hat{\mu}}|_{\mathcal{R}} = \mu$.*

Definition. *Set $E \in C_\sigma(\mathcal{R}(X))$ is $\hat{\mu}$-measurable if*

$$(\forall F \in C_\sigma(\mathcal{R})) :: \hat{\mu}(F) = \hat{\mu}(F \cap E) + \hat{\mu}(F \cap \complement_X E)$$

Note. Set of all $\hat{\mu}$-measurable sets is denoted by $\hat{\mathcal{R}}_\mu$

Theorem. *$\hat{\mathcal{R}}_\mu$ is a σ-ring, and $\hat{\mathcal{R}}_\mu \supset \mathcal{R}_\sigma(X)$.*

Definition. ***Measure on*** *$\hat{\mathcal{R}}_\mu$ is the outer measure $\hat{\mu}$ generated by μ.*

Definition. *Measure on $\hat{\mathcal{R}}_\mu$ is the* ***extension*** *of measure μ on $\mathcal{R}$.*

Definition. ***Measure*** *$\tilde{\hat{\mu}}$* ***on*** *$\mathcal{R}_\sigma(X)$ is the restriction of measure on $\hat{\mathcal{R}}_\mu$.*

Theorem. *Measure on $\mathcal{R}_\sigma(X)$ is unique, i.e. for $\tilde{\hat{\mu}}_1$, $\tilde{\hat{\mu}}_2$ on $\mathcal{R}_\sigma(X)$*

$$(\forall E \in \mathcal{R}) :: \tilde{\hat{\mu}}_1(E) = \tilde{\hat{\mu}}_2(E) \Rightarrow \tilde{\hat{\mu}}_1 = \tilde{\hat{\mu}}_2$$

Note. According to the theorem above, the measure on ring will be from now on automatically extended to the measure on the corresponding σ-ring.

Definition. ***Lebesgue measure*** *m is the usual measure on $\mathcal{L} \equiv \hat{\mathcal{R}}_m(\mathcal{I})$.*

Note. All measures are mathematical structures.

Definition. ***Measurable space*** *$(X, \mathcal{R})$ is a set X together with a ring $\mathcal{R}(X)$.*

Definition. ***Measure space*** *$(X, \mathcal{R}, \mu)$ is a measurable space together with a measure μ on ring $\mathcal{R}$.*

If there is also topology $\mathcal{T}$ on set X, then the measure structure can be made compatible with topological structure through the following way.

Definition. ***Borel σ-algebra*** *$\mathcal{B}_\sigma$ refers to $\mathcal{T}_\sigma(X)$ which is the σ-algebra generated by topology $\mathcal{T}$.*

Definition. ***Borel space*** *refers to $(X, \mathcal{T}_\sigma, \mu)$.*

Definition. *Map $\varphi : X \to Y$ of measurable space $(X, \mathcal{U})$ and $(X', \mathcal{V})$ is* ***measurable*** *if*

$$(\forall E \in \mathcal{V}) :: \varphi^{-1}[E] \in \mathcal{U}$$

Definition. ***Measure isomorphism*** *$\varphi : X \to Y$ is a measurable bijection satisfying $\mathcal{U} = \varphi^{-1}[\mathcal{V}]$*

Note. Hereby, $\varphi^{-1}[\mathcal{V}] := \{\varphi^{-1}[E] \mid E \subset \mathcal{V}\}$

Note. Two measurable spaces are **isomorphic** to each other if there is a measure isomorphism in between.

Theorem. *Measure isomorphism between Borel spaces is continuous.*

Definition. ***Measure-preserving transformation*** *$\varphi : X \to Y$ of space $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{V}, \nu)$ is a map satisfying*
$(\forall E \in \mathcal{U}, F \in \mathcal{V}) ::$
(i) $\varphi[E] \in \mathcal{V}, \quad \varphi^{-1}[F] \in \mathcal{U}$
(ii) $\nu(\varphi[E]) = \mu(E), \quad \mu(\varphi^{-1}[F]) = \nu(F)$

3.1.3 Product Measure

Definition. ***Cartesian product*** *$\mathcal{F} \times \mathcal{G}$ of family of sets $\mathcal{F}$ and $\mathcal{G}$ is defined as*

$$\mathcal{F} \times \mathcal{G} := \{F \times G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$$

Definition. ***Product measurable space*** *of $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ refers to $(X \times Y, \mathcal{R}(\mathcal{U} \times \mathcal{V}))$.*

Note. $\mathcal{U} \times \mathcal{V} := \{A \times B \mid A \in \mathcal{U}, B \in \mathcal{V}\}$.

Definition. ***Product measure*** *$\mu \times \nu$ on $(X \times Y, \mathcal{R}(\mathcal{U} \times \mathcal{V}))$ of $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{V}, \nu)$ is defined as*

$$\mu \times \nu(E) := \sum_{i \in \mathbb{Z}_n} \mu(X_i) \cdot \nu(Y_i)$$

where $E \in \mathcal{R}(\mathcal{U} \times \mathcal{V})$ and $\{X_i \times Y_i\}$ is a partition of E with $X_i \in \mathcal{U}$ and $Y_i \in \mathcal{V}$.

Note. In extending the product measure to the corresponding σ-ring, there is

$$0 \cdot \infty := 0$$

Note. The above relation is defined solely in product measure.

Definition. ***Product measure space*** *is a product measurable space together with product measure.*

Definition. ***Usual measure on*** *$\mathbb{R}^n$ is the product measure of usual measure on $\mathbb{R}$.*

3.2 Integral of Measurable Function

This section deals with measurable functions and their integration.

3.2.1 Lebesgue Integral

Definition. *Set function f on $E \subset X$ of measurable space $(X, \mathcal{R})$ is* ***measurable*** *if*

$$(\forall x, y) :: f^{-1}[x, y) \in \mathcal{R}$$

Definition. *Set $E \subset X$ of space $(X, \mathcal{R}, \mu)$ is a* ***measurable set*** *if $X \in \mathcal{R}$.*

Definition. *Measurable set E is* ***finite*** *if $\mu(E) \in \mathbb{R}$.*

Definition. *Measurable set E is* ***σ-finite*** *if*

$$E \subset \bigcup_{i \in \mathbb{Z}_+} E_i, \;\; \mu(E_i) \in \mathbb{R}$$

Definition. *Set function f is* ***bounded*** *if*

$$(\exists(l, u)) :: \operatorname{img}(f) \subset (l, u)$$

Integral of Bounded Function with Finite Measure

Condition. Let $(X, \mathcal{R}, \mu)$ be a measure space, $E \in \mathcal{R}$ and $\mu(E) \in \mathbb{R}$, f be a bounded set function on E with $\operatorname{img}(f) \subset (l, u)$.

Definition. *Function f is **Lebesgue integrable** if*

$$(\exists S \in \mathbb{R}) :: (\forall \delta > 0, \exists \varepsilon > 0) :: \\ \max\{\mu([x_i, x_{i+1}))\} < \delta \Rightarrow \{\sum_{i \in \mathbb{Z}_n} \xi_i \mu(f^{-1}[x_i, x_{i+1})) - S\}^2 < \varepsilon$$

where $\{[x_i, x_{i+1})\}$ is a partition of $[l, u]$ with cardinality n, and $x_i < \xi_i < x_{i+1}$.

Definition. ***Lebesgue integral*** *of f on E with μ is the number S in the above definition, i.e.*

$$\int_E f \, \mathrm{d}\mu = S$$

Theorem. *All bounded measurable function under the above condition is integrable.*

Integral of Function with σ-Finite Measure

Condition. Let $(X, \mathcal{R}, \mu)$ be a measure space, $E \in \mathcal{R}$ is σ-finite.

Definition. ***Maximum*** *and* ***minimum*** *of sequence $\{x_i\}$ is defined as*

$$\max\{x_i\} = x_k :: (\forall x_i) :: x_i \leqslant x_k$$
$$\min\{x_i\} = x_k :: (\forall x_i) :: x_i \geqslant x_k$$

Definition. ***Positive part*** *f^+ and* ***negative part*** *f^- of function f are functions satisfying*

$$f^+ := \max\{f, 0\}, \quad f^- := \max\{-f, 0\}$$

Theorem. *All functions satisfy $f = f^+ - f^-$.*

Definition. ***Monotonic covering with finite measure*** *of set E is a sequence $\{E_i\}$ with $\mu(E_i) \in \mathbb{R}$ satisfying*

$$E_i \subset E_{i+1}, \quad E = \bigcup_{i \in \mathbb{Z}_+} E_i$$

Definition. ***Lebesgue integral*** *of non-negative function f on E is defined as*

$$\int_E f \, \mathrm{d}\mu := \lim_{n \to \infty} \int_{E_n} [f]_n \, \mathrm{d}\mu$$

where non-negative refers to $f[E] \subset [0, +\infty)$, $\{E_n\}$ is a monotonic covering with finite measure and $[f]_n \equiv$ $\min\{f, n\}$.

Theorem. *Lebesgue integral is irrelevant to the monotonic covering with finite measure.*

Definition. ***Lebesgue integral*** *of function f on E is defined as*

$$\int_E f \, \mathrm{d}\mu := \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu$$

Integration by Substitution

Theorem. *Let φ be measure-preserving transformation between space (X, μ) and (Y, ν). That function f on $E \subset X$ is integrable is equivalent to that function $f \circ \varphi$ is integrable, and*

$$\int_E f(y) \, \mathrm{d}\nu(y) = \int_{\varphi^{-1}[E]} f(x) \, \mathrm{d}\mu(x)$$

Note. "Be equivalent to" refers to "be the necessity and sufficiency of".

3.2.2 Multiple Integral

Condition. Let $(X \times Y, \mathcal{R}(\mathcal{U} \times \mathcal{V}), \mu \times \nu)$ be product space, $E = A \times B \in \mathcal{U} \times \mathcal{V}$, $B_0 \subset B$ and $\nu(B_0) = 0$.

Definition. ***Multiple integral*** *of function $f(x, y)$ integrable on E is defined as*

$$\int_E f(x, y) \, \mathrm{d}\mu \times \nu(x, y)$$

Definition. *Predicate $\psi(f)$ of function f on E of space (X, μ) is true* ***almost everywhere*** *if*

$$(\exists E_0, \mu(E_0) = 0) :: \psi(f|_{E \cup \complement_X E_0}) = true$$

where $f|_{E \cup \complement_X E_0}$ is the restriction of f to $E \cup \complement_X E_0$.

Definition. *Function $f(x, y)$ on $A \times B$ is* ***double integrable*** *if there is an integrable function $h(y)$ on B satisfying*

$$h(y) = \int_A f(x, y) \, \mathrm{d}\mu(x)$$

is true almost everywhere on $B - B_0$.

Definition. ***Double integral*** *of double integrable function $f(x, y)$ is defined as*

$$\int_E f(x, y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) = \int_B h(y) \, \mathrm{d}\nu(y)$$

Theorem. *The following predicates are equivalent:*

(i) *Function f is integrable on E.*
(ii) *Function f is double integrable on $A \times B$.*
(iii) *Function f is double integrable on $B \times A$.*

and

$$\int_E f \, \mathrm{d}\mu \times \nu = \int_E f \, \mathrm{d}\mu \, \mathrm{d}\nu = \int_E f \, \mathrm{d}\nu \, \mathrm{d}\mu$$

4 Real & Complex Analysis

This chapter concerns analysis of number sets.

4.1 Exponentiation

Definition. ***Absolute value*** *of real number x is defined as*

$$|x| = \max\{x, 0\} + \max\{-x, 0\}$$

Note. Real number x is **positive** if $x > 0$.
Note. Real number x is **negative** if $x < 0$.

Definition. ***Cauchy sequence*** *of rational number is a sequence $\{c_i\}$ with $c_i \in \mathbb{Q}$ satisfying*

$$(\forall \varepsilon > 0, \exists l > 0) :: (\forall n, m > l) :: |c_n - c_m| < \varepsilon$$

Theorem. *Every real number can be a limit of some Cauchy sequence.*

Definition. ***Power*** *of $x \in \mathbb{R}$ to positive integer n is defined as*

$$x^n := x \cdot x^{n-1}, \quad x^0 := 1$$

Definition. ***Power*** *of $x \in \mathbb{R}$ to negative integer n is defined as*

$$x^{-n} := (x^n)^{-1}$$

Note. The above exponentiation is defined for all real numbers. However, the following exponentiation is valid only for positive real numbers.

Definition. ***Power*** *of $x \in \mathbb{R}_+$ to rational number $1/n$ is defined as*

$$x^{\frac{1}{n}} := \sup\{y \in \mathbb{R} \mid |y \geqslant 0| \wedge |y^n \leqslant x|\}$$

Note. $\mathbb{R}_+ \equiv (0, +\infty)$.

Definition. ***Power*** *of $x \in \mathbb{R}_+$ to rational number $1 = m/n$ is defined as*

$$x^q := (x^{\frac{1}{n}})^m$$

Definition. ***Power*** *of $x \in \mathbb{R}_+$ to real number α is defined as*

$$x^\alpha := \lim_{i\to\infty} x^{q_i}, \quad \lim_{i\to\infty} q_i = \alpha$$

Theorem. *Power x^α is unique, i.e. for Cauchy sequences*

$$\lim_{i\to\infty} q_i = \alpha = \lim_{i\to\infty} q_i' = \alpha'$$

the corresponding power $x^\alpha = x^{\alpha'}$.

4.2 Differential Calculus

This section elaborates differential calculus. Default function is function of n variable.

4.2.1 Limit and Derivative

Definition. ***Function*** *is a map to a number set.*

Note. Set of all function between set X and Y is denoted by $\mathcal{F}(X, Y)$ with Y being Cartesian product of number field.

Definition. ***Real-valued*** *function is a map $f : X \to \mathbb{R}$.*

Definition. *The binary operation φ already defined on set Y is defined* ***pointwise*** *for map f, g of X and Y if*

$$[f\varphi g](x) := f(x)\,\varphi\, g(x)$$

Note. The addition and multiplication of $\mathcal{F}(X, \mathbb{R})$ is defined pointwise.

Definition. ***Function of*** n ***variable*** *(*n***-ary function****) f is a map $f : \tilde{\mathbb{R}}^n \to \mathbb{R}$ where $\tilde{\mathbb{R}}^n \subset \mathbb{R}^n$.*

Note. Function $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ can be regarded as m functions of n variable. Thus, properties of f is the same as that of these functions.

Definition. ***Limit*** *of function $f(x)$ at point x_0 is defined as*

$$\lim_{x\to x_0} f = q \Leftrightarrow (\exists q \in \mathbb{R}) :: (\forall \{x_i\}, \lim_{i\to\infty} x_i = x_0) :: \lim_{i\to\infty} f(x_i) = q$$

where $\{x_i\}$ is a sequence of $\mathbb{R}^n$.

Definition. ***Kronecker delta*** *${\delta_i}^j$ is a map $\delta : I \times J \to \{0, 1\}$ defined as*

$$i = j \Rightarrow {\delta_i}^j = 1, \quad i \neq j \Rightarrow {\delta_i}^j = 0$$

Definition. ***Partial derivative*** *of function $f(x)$ with respect to coordinate x^i at point x_0 is defined as*

$$\left.\frac{\partial}{\partial x^i} f\right|_{x_0} := \lim_{h\to 0} \frac{1}{h}(f(x + \delta(h)_i) - f(x))$$

where $\delta(h)_i$ is a point in $\mathbb{R}^n$ defined as

$$\delta(h)_i{}^j = h{\delta_i}^j$$

Note. For function of one variable, the partial derivative degenerates to ordinary derivative

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} f\right|_{x_0} \equiv \left.\frac{\partial}{\partial x} f\right|_{x_0}$$

Theorem. *The partial derivative of composite function $f \circ g$ satisfy*

$$\frac{\partial f \circ g}{\partial x^i} = \sum_{j \in \mathbb{Z}_n} \frac{\partial f(u)}{\partial u^j} \frac{\partial g^j(x)}{\partial x^i}$$

where $u = g(x)$.

Note. If the partial derivative of function f exists at every point in domain, it can be overloaded as a **derivative function**

$$\frac{\partial}{\partial x^i} f(y) := \left. \frac{\partial}{\partial x^i} f \right|_y$$

Note. The partial derivative can be overloaded as a map $\partial/\partial x^i : \mathcal{F}(\tilde{\mathbb{R}}^n, \mathbb{R}) \to \mathcal{F}(\tilde{\mathbb{R}}^n, \mathbb{R})$.

Definition. *Function f is **continuously differentiable** if all its partial derivatives exist and are continuous.*

Note. If the partial derivative of f is differentiable, it can also have partial derivative. Usually, if valid, regarding derivative as a map, there is derivative of n order

$$\prod_{i \in \mathbb{Z}_n} \left(\frac{\partial}{\partial x^i} \right)^{m_i} f := \prod_{i \in \mathbb{Z}_n} \prod_{l \in \mathbb{Z}_{m_i}} \left(\frac{\partial}{\partial x^i} \right)^l f$$

The multiplication should be understood as composite of maps.

4.2.2 Special Functions

Definition. *n **times chain product** of a sequence $\{x_i\}$ is defined as*

$$\prod_{i \in \mathbb{Z}_n} x_i := x_n \prod_{i \in \mathbb{Z}_{n-1}} x_i \, , \; \prod_{i \in \mathbb{Z}_1} x_i = x_1$$

Definition. ***Factorial** $n!$ is defined as*

$$n! := \prod_{i \in \mathbb{Z}_n} i$$

Definition. *Constant e is defined as*

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

Definition. *Constant π is defined as*

$$\pi := \int_{[-1,1]} (1 - x^2)^{-\frac{1}{2}} \, \mathrm{d}x$$

Definition. *Function $\sin(x)$ is defined as*

$$\sin(x) := x + \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Note. Function $\cos(x)$ is the derivative of $\sin(x)$.

Definition. *Function $\exp\{x\}$ is defined as*

$$\exp\{x\} := 1 + \sum_{n \in \mathbb{Z}_+} \frac{x^n}{n!}$$

Note. Function $\ln(x)$ is the inverse function of $\exp\{x\}$.

4.2.3 Taylor Expansion

Definition. *Function f is $\mathrm{C}^r, r \in \mathbb{Z}_+$ if it is continuously differentiable up to order r, specially, C^0 refers to continuous, C^∞ refers to $(\forall k \in \mathbb{N}) :: \mathrm{C}^k$.*

Note. Function f is **smooth** if it is C^∞.

The next theorem is on the **Taylor expansion** of smooth function.

Theorem. *Any smooth function f can be expanded as*

$$f(x) = f(a) + \prod_{i \in \mathbb{Z}_n} \sum_{m_i \in \mathbb{Z}_+} \prod_{j \in \mathbb{Z}_n} \frac{(x^j - a^j)^{m_j}}{m_j!} \prod_{k \in \mathbb{Z}_n} \left(\frac{\partial}{\partial x^k} \right)^{m_k} f$$

The next is about the mean value theorem.

Theorem. *Any C^1 function defined on an open connected set satisfy*

$$f(y) - f(x) := \sum_{i \in \mathbb{Z}_n} \frac{\partial f}{\partial x^i} [(1-c)x + cy] \cdot (y^i - x^i)$$

4.3 Complex Analysis

4.3.1 Complex Number Field

Definition. ***Complex number set** $\mathbb{C} = (\mathbb{R}^2, +, \cdot)$ is a set with*

(i) Addition

$$(x_1, y_1) + (x_2, y_2) := (x_1 + y_1, x_2 + y_2)$$

(ii) Multiplication

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$$

where $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$.

Note. **Complex number** is the element of complex number set.

Note. For $z = (x, y) \in \mathbb{C}$, **real part** $\mathrm{Re}(z) := x$, imaginary part $\mathrm{Im}(z) := y$.

Theorem. *Complex number set is a field.*

Definition. ***Imaginary unit*** i *satisfy* $\mathrm{i}^2 = -1$.

Note. Utilizing imaginary unit, every $z \in \mathbb{C}$ can be expressed as $z = x + y\,\mathrm{i}$, and follow the ordinary operation in $\mathbb{R}$.

Definition. ***Complex conjugate*** z^* *of* $z = x + y\,\mathrm{i} \in \mathbb{C}$ *is defined as* $z^* := x - y\,\mathrm{i}$.

The next theorem introduces Euler's formula.

Theorem. *There is the following relationship*

$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$$

4.3.2 Complex Functions

Definition. ***Complex-valued function*** f *is a map* $f : X \to \mathbb{C}$.

Note. Every complex-valued function f can be expressed in the form of real part and imaginary part

$$f(x) = u(x) + \mathrm{i}v(x)$$

Note. The set of all complex-valued function is denoted by $\mathcal{F}(X, \mathbb{C})$.

Note. The addition and multiplication of complex-valued function is defined pointwise.

Definition. ***Conjugate function*** *of complex-valued function* f *is defined as*

$$f^*(x) := [f(x)]^*$$

Definition. ***Complex function*** *is a complex-valued function* $f : D \to \mathbb{C}$ *where* $D \subset \mathbb{C}$.

Definition. ***Derivative*** *of complex function* f *at limit point* $z_0 \in \mathrm{dom}(f)$ *is defined as*

$$\frac{\partial}{\partial z} f\Big|_{z_0} := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Note. Complex function is **differentiable** at z_0 if the derivative exists.

Definition. *Complex function* f *is* ***holomorphic*** *if it is differentiable at every point in domain.*

Note. Complex function f is holomorphic on $S \subset \mathrm{dom}(f)$ if there is open set U of $\mathbb{C}$ such that $S \subset U \subset \mathrm{dom}(f)$ and $\tilde{f}|_U$ is holomorphic.

Note. Complex function f is holomorphic on $z_0 \in \mathrm{dom}(f)$ if there is open set U of $\mathbb{C}$ such that $z_0 \in U \subset \mathrm{dom}(f)$ and $\tilde{f}|_U$ is holomorphic.

Note. The derivative of holomorphic complex function can induce a **derivative function** as

$$\frac{\partial}{\partial z} f(\xi) = \frac{\partial}{\partial z} f\Big|_{\xi}$$

The next theorem is on the Cauchy-Riemann condition for the existence of derivative of complex function.

Theorem. *That complex function* $f(x, y) = u(x, y) + v(x, y)\,\mathrm{i}$ *is differentiable at* $z_0 \in \mathrm{int}(\mathrm{dom}(f))$ *is equivalent to that real and imaginary part of* f *is continuously differentiable and satisfy*

$$\frac{\partial u}{\partial x}\Big|_{z_0} = \frac{\partial v}{\partial y}\Big|_{z_0}, \quad \frac{\partial u}{\partial y}\Big|_{z_0} = -\frac{\partial v}{\partial x}\Big|_{z_0}$$

Definition. *Point* z_0 *is a* ***singularity*** *of complex function* f *if* $z_0 \notin \mathrm{dom}(f)$ *or* f *is not holomorphic at* z_0.

Definition. *Point* z_0 *is an* ***isolated singularity*** *of complex function* f *if there is deleted* ε*-open ball* $\mathring{B}(z_0, \varepsilon) \subset \mathrm{dom}(f)$ *and* f *is holomorphic on* $\mathring{B}(z_0, \varepsilon)$.

Note. Deleted ε-open ball refers to $B(z_0, \varepsilon) \cap \complement_X\{z_0\}$

4.3.3 Integral of Complex Functions

Definition. ***Line*** *of* $\mathbb{R}^n$ *is a map* $l : [a, b] \to \mathbb{R}^n$ *and* $a \neq b$.

Note. A curve can be regarded as n real-valued functions $x^i(t)$.

Definition. ***Loop*** *is a curve satisfying* $l(a) = l(b)$.

Definition. *Line* l *is* C^r *if*

(i) All $x^i(t)$ *is* C^r

(ii) If it is a loop, there is

$$(\forall i \in \mathbb{Z}_n, k \in \mathbb{Z}_r) \lim_{t \to a^+} x^i(t) = \lim_{t \to b^-} x^i(t)$$

Note. Partial limit

$$\lim_{x \to x_0^+} f(x) \ \ (resp \lim_{x \to x_0^+} f(x))$$

is to add condition $x_i > x_0$ (*resp* $x_i < x_0$) to the sequence used to define limit.

Note. Line l is **simple** if

$$(\forall t, t' \in [a, b]) :: l(t) = l(t') \Rightarrow |t = t'| \vee |\{t, t'\} \subset \partial[a, b]|$$

Definition. ***Line integral*** *of $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ along line l is defined as*

$$\int_l f(x)\,\mathrm{d}x^i := \int_{[a,b]} f(x(t))\frac{\mathrm{d}x^i}{\mathrm{d}t}(t)\,\mathrm{d}t$$

Definition. ***Line integral*** *of $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$ along line l is defined as*

$$\int_l f\,\mathrm{d}z := \int_l u\,\mathrm{d}x - v\,\mathrm{d}y + \mathrm{i}\int_l v\,\mathrm{d}x + u\,\mathrm{d}y$$

Note. **Loop integral** is a line integral with the line being a loop.

Definition. ***Residue*** *of complex function f at isolated singularity $z_0 \in \mathrm{cls}(\mathrm{dom}(f))$ is defined as*

$$\mathrm{Res}(f, z_0) := R :: (\exists F) :: \frac{\partial}{\partial z}F(\xi) = f(\xi) - \frac{R}{\xi - z_0}$$

where $F(z)$ is a complex function on $\mathring{B}(z_0, \varepsilon)$.

Note. Euler's formula allows to express every complex number as $z = \rho e^{\mathrm{i}\alpha}$ with **modulus** $\rho = z^*z$ and **argument** α.

Definition. ***Principal argument*** $\mathrm{Arg}(z)$ *of complex number $z \neq 0$ is defined as*

$$\mathrm{Arg}(z) = \alpha :: |\alpha \in (-\pi, \pi]| \wedge |z = \rho e^{\mathrm{i}\alpha}|$$

Definition. *Line l is **counter-clockwise** if*

$$(\forall t_0 \in [a, b]) :: \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \mathrm{Arg}(l(t)) > 0$$

Definition. *Set U is **enclosed** by loop l if $l = \partial U$.*

Definition. *Set $U \subset \mathbb{C}$ enclosed by loop l is **simply connected** if both U and $\complement_{\mathbb{C}}U$ are connected.*

The next theorem elucidates the residue theorem.

Theorem. *For complex function f on $U \cap \complement_{\mathbb{C}}S$ with $U \subset \mathbb{C}$ being a simply connected open set and S being the set of singularities of f, smooth line l in $U \cap \complement_{\mathbb{C}}S$ enclosing $n \in \mathbb{Z}_+$ singularities $\{p_i\}_n$, there is*

$$\oint_l f(z)\,\mathrm{d}z = 2\pi\,\mathrm{i}\sum_{i\in\mathbb{Z}_n} \mathrm{Res}(f, p_i)$$

4.3.4 Fourier Transform

Definition. ***Fourier transform*** *of $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ of x^i is defined as*

$$\mathscr{F}_i[f](k) \equiv \breve{f}_i(k) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{\mathrm{i}k_ix^i}\,\mathrm{d}x^i$$

Note. The integral range is omitted if it integrates over the domain.

Definition. ***Spectrum*** *of $f(x) \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ of x^i is the Fourier transform $\breve{f}_i(k)$.*

Definition. ***Fourier inverse transform*** *of spectrum $\breve{f}(k)$ is defined as*

$$\mathscr{F}_i^{-1}[f](x) \equiv f(x) = \frac{1}{\sqrt{2\pi}}\int \breve{f}_i(k)e^{\mathrm{i}k_ix^i}\,\mathrm{d}k_i$$

5 Functional Analysis

This chapter elaborates linear algebra, metric space and non-linear functional analysis.

5.1 Linear Space

This section introduced basics of linear space. Default space is linear space.

Definition. ***Linear space*** *V over field F is a non-empty set with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ satisfying*

(i) Addition Property $(V, +)$ is an abelian group

(ii) Identity Property

$$(\forall v \in V) :: e \cdot v = v$$

where e is the identity element of F

(iii) Associative Property

$$(\forall \kappa\lambda \in F, v \in V) :: (\kappa\lambda)v = \kappa(\lambda v)$$

(iv) Distributive Property

$$(\forall \kappa \in F, u, v \in V) :: \kappa \cdot (u + v) = \kappa \cdot u + \kappa \cdot v$$
$$(\forall \kappa, \lambda \in F, v \in V) :: (\kappa + \lambda) \cdot v = \kappa \cdot v + \lambda \cdot v$$

Note. Linear space is a mathematical structure (linear structure).

Note. Linear space is also phrased vector space.

Note. Usually, the product symbol $\cdot$ is omitted.

Note. **Real (*resp* complex) vector space** is a vector space over $\mathbb{R}$ (*resp* $\mathbb{C}$).

Note. Set $\mathbb{R}^n$ forms a linear space with addition as addition and scalar multiplication as multiplication.

Definition. ***Vector*** *is the element of a vector space.*

Definition. ***Linear subspace*** *U of space V is a subset of V satisfying*

(i) Additive Identity $0 \in U$

(ii) Addition $(\forall u, v \in U) :: u + v \in U$

(iii) Scalar multiplication $(\forall \kappa \in F, u \in U) :: \kappa u \in U$

Note. Condition q is **unique** for predicate $\psi|_q$ if $\psi|_q = \psi_{q'} = \text{true} \Rightarrow q = q'$.

Definition. ***Direct sum*** *of subspaces U_j, $j \in J$ is a space defined as*

$$(\forall v \in \bigoplus_{j \in J} U_j, \exists! u_j \in U_j) :: v = \sum_{j \in J} u_j$$

Note. Symbol $\exists!$ refers to "$\exists$ unique".

Definition. ***List*** *$\{x_i\}_n$ of set X is a finite subset of sequence $\{x_i\}$ with cardinality being n.*

Definition. *List $\{v_i\}_n$ of space V is* ***linearly independent*** *if*

$$\kappa^i v_i = 0 \Rightarrow \kappa^i = 0 \quad \textit{where} \quad \kappa^i \in F$$

Note. Here, Einstein's summation convention has been introduced, i.e.

$$x_j y^j \equiv \sum_{j \in J} x_j y^j$$

Definition. ***Linear basis*** *of a vector space V is a linearly independent list $\{e_i\}_n$ satisfying*

$$(\forall v \in V) :: v = \sum_{i \in \mathbb{Z}_n} \kappa^i e_i$$

Definition. *Element $\kappa^i \in F$ is the* ***coefficient*** *of vector v on basis $\{e_i\}$.*

Note. The coefficient of vector v can be overloaded as a projection map $\kappa^i : v \mapsto \kappa^i$.

Definition. ***Dimension*** *of vector space V is defined as the cardinality of its linear basis, i.e.*

$$\dim V = \text{card}(\{e_i\}_n)$$

Definition. *List $\{v_i\}_n$* ***spans*** *space V if it is a linear basis of V, i.e.*

$$V = \text{span}\{v_i\}_n$$

Definition. *Space V is* ***finite dimensional*** *if* $\dim V \in \mathbb{Z}_+$. *Space V is* ***infinite dimensional*** *if it is not finite dimensional.*

5.2 Linear Map

This section concerns the linear map between linear spaces and its corresponding matrix. Default space is finite dimensional linear space. Default map is linear map.

5.2.1 Linear Operator

Definition. ***Linear map*** *between space V and V' of field F is a map $T : V \to U$ satisfying*

(i) Additivity $(\forall v, u \in V) :: T(v + u) = T(v) + T(u)$

(ii) Homogeneity $(\forall v \in V, \kappa \in F) :: T(\kappa v) = \kappa T(v)$

Note. Linear map is also phrased linear operator.

Definition. ***Linear isomorphism*** *of space V and V' is a bijective linear map.*

Definition. *Space V and V' are* ***isomorphic*** *if there exists a linear isomorphism in between.*

Note. The set of all linear map between space V and W is denoted by $\mathcal{L}(V, W)$.

Note. Set $\mathcal{L}(V, V)$ is denoted by $\text{End}(V)$, element of $\text{End}(V)$ is the **endomorphism** of space V.

Note. Linear map T is an **automorphism** if it is both an isomorphism and an endomorphism. The set of all automorphism is denoted by $\text{Aut}(V)$.

Definition. ***Addition*** *of map T and S on space V is defined pointwise as*

$$(T + S)(v) := T(v) + S(v)$$

Definition. ***Scalar multiplication*** *of map T on space V is defined pointwise as*

$$(\kappa T)(v) := \kappa(T(v))$$

Definition. ***Composite*** *of map T and S on space V is defined as*

$$TS(v) = T \circ S(v)$$

Definition. *Operator $T \in \mathcal{L}(V, W)$ is* ***invertible*** *if*

$$(\exists S \in \mathcal{L}(W, V)) :: TS = \text{id}_V$$

Theorem. *Operator S in the above definition is unique.*

Proof. If S, S' are the matrix satisfying $TS = TS' = \text{id}_V$, then

$$S = S\text{id}_V = S(TS') = (ST)S' = \text{id}_V S' = S'$$

which ends the proof. □

Definition. ***Inverse*** T^{-1} *of an invertible operator is an operator satisfying* $T^{-1}T = \mathrm{id}_V$.

Definition. ***Eigenvalue*** *and* ***Eigenvector*** *of operator* $T \in \mathrm{End}(V)$ *is* $\lambda \in F$ *and* $v \in V$ *satisfying*

$$(\exists v \in V) :: (T - \lambda \mathrm{id}_V)v = 0$$

5.2.2 Matrix

Definition. $m \times n$ ***Matrix*** (a^i_j) *of field* F *is a list of* F *indexed by* I, J *with cardinality being* m *and* n *respectively.*

Note. **Entry** is the element of a matrix.

Definition. ***Matrix*** (t^i_j) *of linear map* T *between space* V *and* V' *is a list of field* F

$$(t^i_j) = \kappa'^i(T(e_j))$$

where $\{e_j\}$ *is the basis of* V *and* κ'^i *is the projection map under basis of* V'.

Definition. ***Square matrix*** *of order* n *is a* $n \times n$ *matrix.*

Definition. ***Diagonal matrix*** *is a square matrix satisfying*

$$(\forall i \neq j) :: a^i_j = 0$$

Definition. ***Identity matrix*** $\delta^i{}_j$ *is the matrix of identity map.*

Definition. ***Transpose*** *of a square matrix* (a^i_j) *is a matrix*

$$\mathrm{tsp}(a)^i_j = a^j_i$$

5.3 Trace and Determinant

Definition. ***Permutation*** *of* $\mathbb{Z}_n$ *is a bijection* $\sigma : \mathbb{Z}_n \to \mathbb{Z}_n$.

Definition. ***Inversions*** *of permutation* σ *is defined as*

$$\mathrm{inv}(\sigma) := \mathrm{card}(\{ (\sigma_i, \sigma_j) \mid |\sigma_i > \sigma_j| \wedge |i < j| \})$$

Definition. ***Signature*** *of permutation* σ *is defined as*

$$\mathrm{sgn}(\sigma) := (-1)^{\mathrm{inv}(\sigma)}$$

Definition. ***Determinant*** *of square matrix* $A = (a^i_j)$ *of order* n *is defined as*

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i \in \mathbb{Z}_n} a^i_{\sigma(i)}$$

where S_n *is the set of all permutation of* $\mathbb{Z}_n$.

Note. The sum or multiplication over indexing set J is defined as the sum over any bijective list or sequence of J, i.e.

$$\sum_{x \in J} \varphi_x := \sum_{i \in \mathbb{Z}_\delta} \varphi_{x_i}, \quad \prod_{x \in J} \varphi_x := \prod_{i \in \mathbb{Z}_\delta} \varphi_{x_i}$$

where δ can be n or $+$.

Definition. ***Trace*** *of matrix* $A = (a^i_j)$ *is defined as*

$$\mathrm{tr}(A) = \sum_{i \in \mathbb{Z}_n} a^i{}_i$$

Theorem. *If the eigenvalue* $\{\lambda_i\}$ *of matrix* T *over field* F *satisfy* $(\forall \lambda_i) :: \lambda_i \in F$, *there will be*

$$\det(T) = \prod_{\lambda_i \in \{\lambda_i\}} \lambda_i, \quad \mathrm{tr}(T) = \sum_{\lambda_i \in \{\lambda_i\}} \lambda_i$$

5.4 Dual Space

This section concerns the dual space of vector space. Default specific index takes values in $\mathbb{Z}_n$ where n is the dimension.

Definition. ***Dual vector*** v^* *of vector space* V *over field* F *is a linear map* $v^* : V \to F$.

Definition. ***Dual (vector) space*** *of vector space* V *is the set of all dual vectors of* V.

Definition. *Dual space of* V *over* F *becomes a vector space under*

(i) ***Addition***

$$(\omega_1 + \omega_2)(v) := \omega_1(v) + \omega_2(v)$$

(ii) ***Scalar multiplication***

$$(\kappa\omega)(v) := \kappa(\omega(v))$$

where $\omega, \omega_1, \omega_2 \in V^*, v \in V, \kappa \in F$

Theorem. $\dim(V^*) = \dim(V)$.

Proof. Define list $\{e^{*a}\}$ as $e^{*a}(e_b) := \delta^a{}_b$ where $\{e_b\}$ is the basis of V and $\delta^a{}_b$ is the identity matrix. There is

(i) List $\{e^{*a}\}$ is linearly independent since

$$\kappa_a e^{*a} = 0 \Rightarrow \kappa_a e^{*a}(e_b) = \kappa_a \delta^a{}_b = \kappa_b = 0$$

(ii) Any $v \in V$ can be expressed as $v = v^a e_a$ and e_a satisfy

$$\omega(e_a) = \omega(e_b)\delta^b{}_a = \omega_b e^{*b}(e_a)$$

where $\omega_b \equiv \omega(e_b) \in F$. Thus, $\omega = \omega_b e^{*b}$.

(iii) $\mathrm{card}(\{e^{*a}\}) = n$.

Thus, the theorem is proved. □

Definition. ***Dual basis*** *of V^* is the list $\{e^{*a}\}$ in the above proof.*

Definition. ***Coefficient*** *of ω under basis $\{e^{*a}\}$ is ω_b in the above proof.*

Theorem. *There exists a natural isomorphism between vector space V and its double dual space V^{**}.*

Note. Due to the theorem above, double and even higher order dual space are not specified.

5.5 Variation Method

This section introduces metric space and normed linear space as well as constructs the variation operation on Banach space.

5.5.1 Metric Space

Definition. ***Distance*** *of a non-empty set R is a map $\rho : R \times R \to \mathbb{R}$ satisfying*

(i) Positive definite $\rho(x, y) \geqslant 0$
(ii) Non-degenerate $\rho(x, y) = 0 \Leftrightarrow x = y$
(iii) Triangle inequality $\rho(x, y) \leqslant \rho(x, z) + \rho(y, z)$

Definition. ***Metric space*** *(R, ρ) is a set R with the distance ρ of R.*

Note. Distance is a mathematical structure.

Note. $\mathbb{R}^n$ becomes a metric space under the distance of $\mathbb{R}^n$.

Definition. ***ε-open ball*** *of point p in metric space is defined as*

$$B(p, \varepsilon) := \{\, q \mid \rho(p, q) < \varepsilon \,\}$$

where $\varepsilon \in \mathbb{R}$.

Definition. ***Induced topology of metric space*** *$\mathcal{T}_i$ is the topology generated by subbasis being the set of all open balls.*

Note. Metric space becomes a topological space under induced topology.

5.5.2 Normed Linear Space

Definition. ***Semi-norm*** *of linear space V over field $F \in \{\mathbb{R}, \mathbb{C}\}$ is a map $\|\cdot\| : V \to F$ satisfying*

(i) Positive definite $\|x\| \geqslant 0$
(ii) Homogeneity $\|\alpha x\| = |\alpha| \cdot \|x\|$
(iii) Triangle inequality $\|x + y\| \leqslant \|x\| + \|y\|$

Note. $|\alpha|$ for $\alpha \in \mathbb{C}$ is defined as $\sqrt{\alpha^* \alpha}$.

Definition. ***Norm*** *is a semi-norm satisfying*

$$\|x\| = 0 \Leftrightarrow x = 0$$

Definition. ***Normed linear space*** *$(V, \|\cdot\|)$ is a linear space V with a norm $\|\cdot\|$.*

Note. Norm is a mathematical structure.

Definition. ***Induced distance*** *of normed linear space is defined as*

$$\rho(x, y) := \|x - y\|$$

Note. Normed linear space becomes a metric space under induced distance.

Definition. ***Cauchy sequence*** *in metric space (R, ρ) is a sequence $\{c_i\}$ satisfying*

$$(\forall \varepsilon > 0, \exists l > 0) :: (\forall n, m > l) :: \rho(c_n, c_m) < \varepsilon$$

Definition. *Metric space is* ***complete*** *if all Cauchy sequence converges to point within space.*

Definition. ***Banach space*** *is a complete normed linear space.*

5.5.3 Variation on Banach Space

Definition. ***Functional*** *is a map $\mathcal{F} : V \to F$ where V is a linear space and F is real or complex field.*

Condition. E and $F \in \{\mathbb{R}, \mathbb{C}\}$ are Banach spaces, $D \subset E$ is open, $\mathcal{F} : D \to F$, $\eta, \varphi \in E$, $\varepsilon \in F$.

Definition. *Functional $\mathcal{F}$ is* ***Gâteaux differentiable*** *if*

$$(\forall \eta \in E) :: \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\mathcal{F}[\varphi + \varepsilon \eta] - \mathcal{F}[\varphi]) \equiv \frac{\delta \mathcal{F}}{\delta \varphi} \in F$$

Definition. ***Variation*** *$\delta \mathcal{F}$ of functional $\mathcal{F}$ is defined as*

$$\delta \mathcal{F} = \frac{\delta \mathcal{F}}{\delta \varphi} \delta \varphi$$

where $\delta \varphi \equiv \varepsilon \eta$.

5.6 Inner Product Space

Definition. ***Inner product*** $(\cdot,\cdot)$ *of linear space* V *over field* $F \in \{\mathbb{R}, \mathbb{C}\}$ *is a map* $(\cdot,\cdot) : V \times V \to F$ *satisfying*

(i) *Additive* $(f, g + h) = (f, g) + (f, h)$
(ii) *Homogeneity* $(f, cg) = c(f, g)$
(iii) *Conjugate* $(f, g)^* = (g, f)$
(iv) *Non-degenerate* $(f, f) = 0 \Leftrightarrow f = 0$

where $f, g, h \in V, c \in F$.

Definition. *Inner product* $(\cdot,\cdot)$ *is* ***positive-definite*** *if* $(\forall f \in V) :: (f, f) \geqslant 0$.

Definition. ***Inner product space*** *is a linear space* V *with an inner product* $(\cdot,\cdot)$.

Definition. ***Inner product*** $\cdot$ *on* $\mathbb{R}^n$ *is defined as* $x \cdot y = x_i y^i$.

Definition. ***Induced norm*** *on inner product space is defined as*

$$\|f\| := \sqrt{(f, f)}$$

Note. Inner product space becomes a normed linear space under the induced norm.

Note. Inner product $(\cdot,\cdot)$ naturally induced an anti-linear injection $\nu : V \to V^*$ defined as

$$\nu(f) := (f, \cdot)$$

Anti-linear refers to $\nu(cf) = c^* \nu(f)$.

Definition. ***Hilbert space*** *is a complete inner product space.*

Theorem. *The dual space of an inner product space is complete.*

Definition. ***Absolute*** p***th power*** $|f|^p$ $(p \in \mathbb{Z}_+)$ *is a map from* $\mathcal{F}(X, Y)$ *to* $\mathcal{F}(X, F)$ *where* $F \in \{\mathbb{R}, \mathbb{C}\}$.

Note. Absolute square of $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{C})$ is defined as $f^* \cdot f$.

Definition. $L^p[E, \mu]$ ***space*** *is a set of function on measure space* $(E, \mathcal{R}, \mu)$ *satisfying*

$$\left[\int_E |f|^p \,\mathrm{d}\mu\right]^{\frac{1}{p}} \in F$$

Definition. ***Inner product*** *of* $L^2[E, \mu]$ *is defined as*

$$(f, g) := \int_E f^* \cdot g \,\mathrm{d}\mu$$

6 Differential Geometry

This chapter introduces Differential Geometry which assigns differential structure to a topological space. Default topological space is Hausdorff and second-countable. Default manifold is smooth differential manifold. Default maps, functions and fields are smooth.

6.1 Differential Manifold

Definition. n***-dimensional Coordinate system*** (U, ψ) *of topological space* M *is an open set* U *together with a homeomorphism* $\psi : U \to V$ *where* V *is an open set of* $\mathbb{R}^n$.

Note. Coordinate system is also phrased **chart**.

Note. Open set U is the **coordinate patch** of coordinate system (U, ψ).

Note. For $p \in U$, $\psi(p) \in \mathbb{R}^n$ is the **coordinate** of point p.

Definition. ***Coordinate transformation*** *between coordinate system* (U_α, ψ_α) *and* (U_β, ψ_β) *satisfying* $U_\alpha \cap U_\beta$ *is the map* $\psi_\beta \circ \psi_\alpha^{-1}$.

Definition. *Coordinate system* (U_α, ψ_α) *and* (U_β, ψ_β) *satisfying* $U_\alpha \cap U_\beta$ *is* C^k***-compatible*** *if the coordinate transformation in between is* C^k*, denoted by*

$$(U_\alpha, \psi_\alpha) \overset{k}{\sim} (U_\beta, \psi_\beta)$$

Definition. C^k***-atlas*** *is a set of charts*

$$\{(U, \psi)\} := \{(U_\alpha, \psi_\alpha) \mid (U_\alpha, \psi_\alpha)\} \overset{k}{\sim} (U, \psi)\}$$

Definition. n***-dimensional*** C^k ***differential manifold*** M *is a topological space with a* C^k*-atlas.*

Note. C^k-atlas is a mathematical structure — **differential structure**.

Note. Dimension $\dim M = n$.

Definition. *Map* f *between manifolds* M *and* M' *is* C^k *if for chart* (U, ψ) *of* M *and* (U', ψ') *in* M'*, map* $\psi' \circ f \circ \psi^{-1}$ *is* C^k.

Note. Thus, f and $\psi' \circ f \circ \psi^{-1}$ are not distinguished.

Definition. ***Diffeomorphism*** *between manifold* M *and* M' *is a bijection* f *with* f *and* f^{-1} *both smooth.*

Note. Manifold M and M' is **diffeomorphic** if there is a diffeomorphism in between.

Definition. ***Function on*** M *is a map* $f : M \to \mathbb{R}$.

Note. All smooth functions on M forms a set $\mathscr{F}_M$.

Note. Coordinate x^i is a smooth function on M.

6.2 Linear Structure

Definition. ***Multi-linear map*** *between vector space* $\{V_i\}_n$ *and W is a map*

$$f : \prod_{i\in\mathbb{Z}_n} V_i \to W$$

satisfying $f(\cdots v_i \cdots)$ *is linear.*

Note. Define the anti-projection map q_j from X_j to Cartesian product of $\{X_i\}$ as

$$p_j(q_j(x)) = x, \ \ (\forall i \neq j) :: p_i(q_j(x)) = p_i(q_j(y))$$

Then map f on Cartesian product of $\{X_i\}$ has a restriction $f(\cdots x_j \cdots)$ defined as $f \circ q_j$.

Note. In defining anti-projection map q_i, only the ith variable is considered. Thus, if hold the map q_i, it is a bijection X_i to Cartesian product X of $\{X_j\}$; if assign the ith variable, $q_i(x_i)$ can be a point in $\{y \in X \mid p_i(y) = x_i\}$

Definition. ***Tensor of type*** (k, l) *is a linear map*

$$T : \prod_{i\in\mathbb{Z}_k} V_i^* \times \prod_{j\in\mathbb{Z}_l} V_j \to \mathbb{R}$$

Definition. ***Tensor of type*** (k, l) *on vector space* V *is a linear map*

$$T : \prod_{i\in\mathbb{Z}_k} V^* \times \prod_{j\in\mathbb{Z}_l} V \to \mathbb{R}$$

Note. The set of all tensors of type (k, l) is denoted by $\mathscr{T}(k, l)$. The set of all tensors of type (k, l) on vector space V is denoted by $\mathscr{T}_V(k, l)$.

Note. Tensor of type $(0, 0)$ is a scalar $T \in \mathbb{R}$.

Definition. ***Tensor product*** $T \otimes T'$ *of* $T \in \mathscr{T}_V(k, l)$ *and* $T' \in \mathscr{T}_W(k', l')$ *is a tensor of type* $(k + k', l + l')$

$$T \otimes T'(\omega, v, \omega', v') := T(\omega, v)T'(\omega', v')$$

where

$$\omega \in \prod_{i\in\mathbb{Z}_k} V^*, v \in \prod_{j\in\mathbb{Z}_l} V, \ \omega' \in \prod_{i\in\mathbb{Z}_k'} W^*, v' \in \prod_{j\in\mathbb{Z}_l'} W$$

Definition. $\mathscr{T}(k, l)$ *becomes a vector space under* ***addition*** *and* ***scalar multiplication*** *defined pointwise.*

Theorem. $\dim \mathscr{T}_V(k, l) = n^k n^l$

Proof. The following list is a basis of $\mathscr{T}_V(k, l)$

$$\{\bigotimes_{i\in\mathbb{Z}_k} e^i_{a_i} \otimes \bigotimes_{j\in\mathbb{Z}_l} e^{*b_j}_j\}$$

There are in total $n^k n^l$ elements, and thus the theorem is proved. □

Now invoke symbol $a_1 \cdots a_n$ defined iteratively

$$a_1 \cdots a_n := a_1 \cdots a_{n-1} a_n, \ \ a_1 = a_1$$

Next, abstract index notation of tensors is introduced. There are the following symbols for tensors of type (k, l)

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}, \ \ T^{\mu_1\cdots\mu_i\cdots\mu_k}{}_{\nu_1\cdots\nu_j\cdots\nu_l}, \ \ T^{\cdots\mu\cdots}_{\cdots\nu\cdots}$$

The first symbol is used for denoting an ordinary tensor of type (k, l), the second symbol is used in denoting the **action** of index i and j

$$T^{\mu_1\cdots\mu_i\cdots\mu_k}{}_{\nu_1\cdots\nu_j\cdots\nu_l}\omega_{\mu_i}v^{\nu_j} := T(\cdots\omega_i\cdots, \cdots v^j \cdots)$$

where $T(\cdots\omega_i\cdots, \cdots v^j \cdots)$ is defined as $T(q_i(\omega_i), q_{k+j}(v^j))$ with q_i being the anti-projection map.

The third symbol is used to briefly denote the ordinary tensors and full actions, i.e.

$$T^{\cdots\mu\cdots}_{\cdots\nu\cdots}\omega_\mu v^\nu := T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\omega^1_{\mu_1}\cdots\omega^k_{\mu_k}v^{\nu_1}_1\cdots v^{\nu_l}_1$$

Definition. ***Coefficient*** *of tensor* T *under basis* $\{e^a_\mu\}$ *and* $\{e^\nu_b\}$ *is defined as*

$$T^{\cdots a\cdots}_{\cdots b\cdots} := T^{\cdots\mu\cdots}_{\cdots\nu\cdots}e^a_\mu e^\nu_b$$

Definition. ***Contraction*** C^i_j *of tensor* $T \in \mathscr{T}_V(k, l)$ *is a map* $C^i_j : \mathscr{T}_V(k, l) \to \mathscr{T}_V(k-1, l-1)$ *defined as*

$$C^i_j T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} := T^{\mu_1\cdots\mu_i\cdots\mu_k}{}_{\nu_1\cdots\nu_j\cdots\nu_l}e^a_{\mu_i}e^{\nu_j}_a$$

where $e^a_{\mu_i}$ *and* $e^{\nu_j}_a$ *are basis.*

Theorem. *Contraction is irrelevant to the basis selected.*

Note. The action of tensor T is equivalent to the composite of tensor product and contraction, thus they are not distinguished.

Definition. ***Symmetric part*** *of a tensor is defined as*

$$T_{\cdots(\mu_1\cdots\mu_l)\cdots} := \frac{1}{l!}\sum_{\sigma\in S_n} T_{\cdots(\mu_{\sigma(1)}\cdots\mu_{\sigma(l)})\cdots}$$

$$T^{\cdots(\mu_1\cdots\mu_l)\cdots} := \frac{1}{l!}\sum_{\sigma\in S_n} T^{\cdots(\mu_{\sigma(1)}\cdots\mu_{\sigma(l)})\cdots}$$

Note. The above definition defines the symmetry of index $\mu_1 \cdots \mu_l$ of the tensor.

Note. Tensor T is **totally symmetric** if it equals the symmetry of all its indexes.

Definition. ***Antisymmetric part*** *of a tensor is defined as*

$$T_{\cdots[\mu_1\cdots\mu_l]\cdots} := \frac{1}{l!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)T_{\cdots[\mu_{\sigma(1)}\cdots\mu_{\sigma(l)}]\cdots}$$
$$T^{\cdots[\mu_1\cdots\mu_l]\cdots} := \frac{1}{l!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)T^{\cdots[\mu_{\sigma(1)}\cdots\mu_{\sigma(l)}]\cdots}$$

Note. Tensor T is **totally antisymmetric** if it equals the antisymmetry of all its indexes.

Definition. ***Vector*** *at point p in manifold M is a map $v : \mathscr{F}_M \to \mathbb{R}$ satisfying*

(i) *Linearity* $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$

(ii) *Leibniz* $v(fg) = f|_p v(g) + g|_p v(f)$

where $f, g \in \mathscr{F}_M, \alpha, \beta \in \mathbb{R}$.

Definition. *Set of all vectors at p becomes a vector space* T_pM *under*

(i) ***Addition*** $(v_1 + v_2)(f) := v_1(f) + v_2(f)$

(ii) ***Scalar multiplication*** $(\alpha v)(f) := \alpha(v(f))$

where $v_1, v_2 \in \mathrm{T}_pM, f \in \mathscr{F}_M, \alpha \in \mathbb{R}$.

Lemma. *If the restriction of $\hat{f}$ and f on $N \in \mathcal{N}(p)$ are equal, then $v \in \mathrm{T}_pM$ satisfies $v(\hat{f}) = v(f)$.*

Note. The above lemma allows vector to act on any smooth function defined on a subset of M.

Theorem. $\dim(\mathrm{T}_pM) = \dim M$.

Proof. Define list $\{X_a\}$ of T_pM as

$$X_a(f) := \left.\frac{\partial}{\partial x^a}f\right|_p$$

There is

(i) List $\{X_a\}$ is linearly independent since

$$\kappa^a X_a = 0 \Rightarrow \kappa^a X_a(x^b) = \kappa^a\frac{\partial x^b}{\partial x^a} = \kappa^a\delta^b{}_a = 0$$

(ii) Any vector $v \in \mathrm{T}_pM$ can be expressed as $v = v^a X_a$ with $v^a \in \mathbb{R}$.
According to the mean value theorem, $\forall f \in \mathscr{F}_M$, there is

$$f(q) = f(p) + [x^a|_q - x^a|_p]H_a(q)$$

where $H_a(q) = \partial f/\partial x^a[(1-c)p + cq]$ with $c \in \mathbb{R}$. Thus

$$\begin{aligned}v(f) =& v[f(p)] + [x^a|_q - x^a|_p]|_p v(H_a)\\ &+ H_a(q)|_p v[x^a|_q - x^a|_p]\\ =& v(x^a)H_a(p) = v(x^a)X_a(f)\end{aligned}$$

Denote $v(x^a)$ as v^a, there is $v = v^a X_a$.

(iii) $\mathrm{card}\{X_a\} = n$.

Thus, the theorem is proved. □

Note. List $\{X_a\}$ is the **coordinate basis** of T_pM, $X_a \in \{X_b\}$ is a **coordinate basis vector**, coefficient v^μ is the **coordinate components** of vector v.

Definition. ***Dual vector space*** T_p^*M *is the dual space of* T_pM.

Note. **Dual coordinate basis** $\{\mathrm{d}x^a\}$ is the dual basis of coordinate basis.

Definition. ***Tensor of type*** (k, l) *at point $p \in M$ is a tensor of type (k, l) on vector space* T_pM.

Note. Set of all tensors of type (k, l) at p is denoted by $\mathrm{T}_p(k, l)$.

Definition. ***Tensor field*** *(of type (k, l)) on manifold M is a map $T : x \mapsto T|_x$ where $T|_x \in \mathrm{T}_x(k, l)$.*

Note. Tensor field of type $(1, 0)$ is a **vector field**; tensor field of type $(0, 1)$ is a **dual vector field**.

Note. Tensor field is usually overloaded as its image.

Definition. *Vector field v is C^k if map $v(f) : x \mapsto v|_x(f)$ with $f \in \mathscr{F}(M, \mathbb{R})$ is C^k.*

Definition. *Dual vector field ω is C^k if map $\omega(v) : x \mapsto \omega|_x(v|_x)$ is C^k, where v is a smooth vector field.*

Definition. *Tensor field T is C^k if map*

$$T : x \mapsto T|_x{}^{\cdots\mu\cdots}{}_{\cdots\nu\cdots}(\omega|_x)_\mu v|_x^\nu$$

is C^k, where ω^i and v_i are smooth vector field and dual vector field respectively.

Definition. C^k ***curve*** *on manifold M is a map $C : I \to M$, where I is an interval of $\mathbb{R}$.*

Note. **Parameter** of curve $C : I \to M$ is $t \in I$.

Definition. ***Tangent vector*** *of C^1 curve C at $C(t_0)$ is defined as*

$$T(f) := \left.\frac{\mathrm{d}(f\circ C)}{\mathrm{d}t}\right|_{t_0}$$

Definition. ***Integral curve*** *$C(t)$ of vector field v is a curve satisfying $T|_{C(t)} = v|_{C(t)}$ where T is the tangent vector.*

Theorem. *There is a unique inextendible integral curve of smooth vector field passing through point p and satisfying $C(0) = p$.*

Note. Inextendible refers to

$$(\nexists C') :: |\tilde{C'} = C| \wedge |\mathrm{dom}(C) \subsetneq \mathrm{dom}(C')$$

6.3 Metric Structure

Definition. ***Metric*** *$g_{\mu\nu}$ on vector space V is a tensor of type $(0,2)$ satisfying*

(i) Symmetry $g_{\mu\nu} = g_{\nu\mu}$
(ii) Non-degeneracy $(\forall u^\mu \in V) :: g_{\mu\nu}u^\mu v^\nu = 0 \Rightarrow v^\nu = 0$

Note. Metric is a mathematical structure.

Definition. ***Norm induced by metric*** *$g_{\mu\nu}$ is defined as*

$$\|v\| := \sqrt{g_{\mu\nu}v^\mu v^\nu}$$

Definition. ***Orthonormal basis*** *$\{e_a^\mu\}$ is a basis of V satisfying*

(i) Orthogonal $(\forall a \neq b) :: g_{\mu\nu}e_a^\mu e_b^\nu = 0$
(ii) Normal $(\forall a = b) :: |g_{\mu\nu}e_a^\mu e_b^\nu| = 1$

Definition. ***Signature*** *of metric is defined as*

$$\mathrm{sgn}(g_{\mu\nu}) := \mathrm{tr}(g_{\mu\nu}e_a^\mu e_b^\nu)$$

where $\{e_a^\mu\}$ is the orthonormal basis.

Theorem. *Any vector space with metric has an orthonormal basis, and the signature is irrelevant to the basis.*

Definition. *Metric $g_{\mu\nu}$ on M is* ***Positive definite*** *if*

$$\mathrm{sgn}(g_{\mu\nu}) = \dim M$$

Definition. *Metric $g_{\mu\nu}$ on M is* ***Lorentzian*** *if*

$$\mathrm{sgn}(g_{\mu\nu}) = \dim M - 2$$

Definition. *In vector space with Lorentzian metric, vectors are divided into*

(i) ***Time-like*** *$g_{\mu\nu}v^\mu v^\nu < 0$*
(ii) ***Space-like*** *$g_{\mu\nu}v^\mu v^\nu > 0$*
(iii) ***Light-like*** *$g_{\mu\nu}v^\mu v^\nu = 0$*

Note. Metric can naturally induce a map

$$g_{\mu\nu} : v^\mu \mapsto v_\nu \equiv g_{\mu\nu}v^\mu$$

Definition. ***Metric field*** *is a tensor field satisfying*

(i) Symmetry $g_{\mu\nu} = g_{\nu\mu}$
(ii) Non-degeneracy $g_{\mu\nu}u^\mu v^\nu = 0 \Rightarrow v^\nu = 0$
(iii) Signature $\mathrm{sgn}(g_{\mu\nu}|_p) = \mathrm{sgn}(g_{\mu\nu}|_q)$

Note. Curve C is **time-like**, **space-like** or **light-like** if its tangent vector at any point of C is time-like, space-like or light-like.

Definition. ***Generalized Riemann space*** *$(M, g_{\mu\nu})$ is a connected manifold M with a metric $g_{\mu\nu}$.*

Definition. ***Spacetime*** *$(M, g_{\mu\nu})$ is a connected manifold M with a Lorentzian metric $g_{\mu\nu}$.*

Definition. ***Euclidean metric*** *$\delta_{\mu\nu}$ of $\mathbb{R}^n$ is defined as*

$$\delta_{\mu\nu} := \delta_{ab}\,\mathrm{d}x_\mu^a\,\mathrm{d}x_\nu^b$$

where $\mathrm{d}x_\mu^a$ is the dual coordinate basis.

Definition. ***n-dimensional Euclidean space*** *is $(\mathbb{R}^n, \delta_{\mu\nu})$.*

Definition. ***Minkowski metric*** *$\eta_{\mu\nu}$ of $\mathbb{R}^n$ is defined as*

$$\eta_{\mu\nu} := -\delta_a^0\delta_b^0\,\mathrm{d}x_\mu^a\,\mathrm{d}x_\nu^b + \delta_{ij}\,\mathrm{d}x_\mu^i\,\mathrm{d}x_\nu^j$$

where index $i, j \in \mathbb{Z}_{n-1}$.

Definition. ***n-dimensional Minkowski space*** *is $(\mathbb{R}^n, \eta_{\mu\nu})$.*

Definition. *Metric $g_{\mu\nu}$ on M can naturally induce a map $g : \mathrm{T}_xM \to \mathrm{T}_x^*M$ as*

$$v_\mu := g_{\mu\nu}v^\nu$$

Note. The inverse of $g_{\mu\nu}$ is a tensor of type $(2,0)$ denoted by $g^{\mu\nu}$.

Note. This indicates that the metric can be used to exchange the indexes, and this operation is set default in the following context.

6.4 Connection Structure

Note. Smooth tensor field of type (k,l) on M is denoted by $\mathscr{F}_M(k,l)$.

Definition. ***Connection*** *∂_μ on M is a map $\partial_\mu : \mathscr{F}_M(k,l) \to \mathscr{F}_M(k,l+1)$ satisfying*

(i) Linearity $(\forall T^{\cdots\mu\cdots}_{\cdots\nu\cdots}, S^{\cdots\rho\cdots}_{\cdots\sigma\cdots} \in \mathscr{F}_M(k,l)) ::$

$$\partial_\lambda(\alpha T^{\cdots\mu\cdots}_{\cdots\nu\cdots} + \beta S^{\cdots\rho\cdots}_{\cdots\sigma\cdots}) = \alpha\partial_\lambda T^{\cdots\mu\cdots}_{\cdots\nu\cdots} + \beta\partial_\lambda S^{\cdots\rho\cdots}_{\cdots\sigma\cdots}$$

(ii) Leibniz $(\forall T^{\cdots\mu\cdots}_{\cdots\nu\cdots} \in \mathscr{F}_M(k,l),\ S^{\cdots\rho\cdots}_{\cdots\sigma\cdots} \in \mathscr{F}_M(k',l')) ::$

$$\partial_\lambda(T^{\cdots\mu\cdots}_{\cdots\nu\cdots}S^{\cdots\rho\cdots}_{\cdots\sigma\cdots}) = S^{\cdots\rho\cdots}_{\cdots\sigma\cdots}\partial_\lambda T^{\cdots\mu\cdots}_{\cdots\nu\cdots} + T^{\cdots\mu\cdots}_{\cdots\nu\cdots}\partial_\lambda S^{\cdots\rho\cdots}_{\cdots\sigma\cdots}$$

(iii) Commutative with contraction $\partial_\mu \circ C_j^i = C_j^i \circ \partial_\mu$
(iv) Relation to vector $v(f) = v^\mu\partial_\mu f$

Note. Connection is a mathematical structure.

Theorem. *Any manifold has connections.*

Theorem. *For connection $\grave{\partial}_\mu$ and ∂_μ, $\grave{\partial}_\mu f = \partial_\mu f$.*

Lemma. *If the restriction of $\hat{T}, T \in \mathscr{F}_M(k,l)$ on $N \in \mathcal{N}(p)$ are equal, then $\partial_\mu \hat{T} = \partial_\mu T$.*

Definition. ***Ordinary derivative*** *∂_a is defined as*

$$\partial_a T^{\cdots\mu\cdots}_{\cdots\nu\cdots} := \partial_a T^{\cdots b\cdots}_{\cdots c\cdots} X^\mu_a \, \mathrm{d}x^c_\nu$$

Definition. ***Covariant derivative*** *is a connection that is irrelevant to the coordinate system.*

Note. Ordinary derivative is not a covariant derivative.

Definition. *Connection ∂_μ is **compatible with metric** $g_{\mu\nu}$ if*

$$\partial_\rho g_{\mu\nu} = 0$$

Theorem. *There is unique compatible connection on $(M, g_{\mu\nu})$.*

Note. The metric structure and connection structure on the same manifold is asked to be compatible.

Definition. ***Torsion tensor*** *$T^\sigma{}_{\mu\nu}$ of connection ∂_μ is defined by*

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) f = T^\sigma{}_{\mu\nu} \partial_\sigma f$$

where $f \in \mathscr{F}_M$.

Note. Connection ∂_μ is **torsion-free** if its torsion tensor $T^\sigma{}_{\mu\nu} = 0$.

Definition. ***Riemann curvature*** *$R_{\mu\nu\rho}{}^\sigma$ of connection ∂_μ is defined by*

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \omega_\rho = R_{\mu\nu\rho}{}^\sigma \omega_\sigma$$

where $\omega_\rho \in \mathscr{F}_M(0,1)$.

Note. Connection ∂_μ is **flat** if its Riemann curvature $R_{\mu\nu\rho}{}^\sigma = 0$.

Definition. ***Ricci tensor*** *$R_{\mu\nu}$ is defined as $R_{\mu\nu} := R_{\mu\rho\nu}{}^\rho$*

Definition. ***Scalar curvature*** *R is defined as $R := g^{\mu\nu} R_{\mu\nu}$, where $g^{\mu\nu}$ is the inverse of the metric.*

Definition. ***Einstein tensor*** *$G_{\mu\nu}$ is defined as*

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Theorem. *Einstein tensor satisfies*

$$\partial^\mu G_{\mu\nu} = 0$$

Note. The above property is essential in preserving the energy conservation law.

6.5 Spacetime Symmetry

This section deals with the map between manifolds with extra structures.

Pull-back and Push-forward

Condition. Set M, N manifolds, $\varphi : M \to N$ smooth map.

Definition. ***Cotangent map*** *(or **pull-back**) $\mathrm{q}\varphi : \mathcal{F}_N \to \mathcal{F}_M$ is defined as*

$$\mathrm{q}\varphi(f)|_p := f|_{\varphi(p)}$$

Note. Set of all smooth tensor fields on manifold M is denoted by $\mathcal{F}_M(k,l)$.

Note. Overload of a concept is to use the same symbol to represent the extension of the concept.

Definition. ***Tangent map*** *(or **push-forward**) $\mathrm{d}\varphi : \mathrm{T}_p M \to \mathrm{T}_{\varphi(p)} N$ is defined as*

$$\mathrm{d}\varphi(v)(f) := v(\mathrm{q}\varphi(f))$$

Definition. ***Pull-back*** *can be extended and overloaded as $\mathrm{q}\varphi : \mathcal{F}_N(0,l) \to \mathcal{F}_M(0,l)$ with*

$$\mathrm{q}\varphi(T)_{\ldots\mu\ldots} v^\mu := T_{\ldots\mu\ldots} \mathrm{d}\varphi(v)^\mu$$

Definition. ***Push-forward*** *can be extended and overloaded as $\mathrm{d}\varphi : \mathrm{T}_x(k,0) \to \mathrm{T}_{\varphi(x)}(k,0)$ where $x \in M$ with*

$$\mathrm{d}\varphi(T)^{\cdots\mu\cdots} \omega_\mu := T^{\cdots\mu\cdots} \mathrm{q}\varphi(\omega)_\mu$$

Condition. Reset φ a diffeomorphism.

Definition. ***Push-forward*** *of φ can be extended and overloaded as $\mathrm{d}\varphi : \mathcal{F}_M(k,l) \to \mathcal{F}_N(k,l)$ with*

$$\mathrm{d}\varphi(T)^{\cdots\mu\cdots}_{\cdots\nu\cdots} \omega_\mu v^\nu := T^{\cdots\mu\cdots}_{\cdots\nu\cdots} \mathrm{q}\varphi(\omega)_\mu \mathrm{q}\varphi(v)^\nu$$

where $\mathrm{q}\varphi(v)^\nu$ is the overload of $\mathrm{d}(\varphi^{-1})(v)^\nu$.

Definition. ***Pull-back*** *of φ can be extended and overloaded as the inverse of its push-forward.*

Theorem. *Pull-back and push-forward are linear map.*

Theorem. *Pull-back and push-forward is commutative with tensor product, i.e.*

$$\mathrm{d}\varphi(TS)^{\cdots\mu\cdots\rho\cdots}_{\cdots\nu\cdots\sigma\cdots} = \mathrm{d}\varphi(T)^{\cdots\mu\cdots}_{\cdots\nu\cdots} \mathrm{d}\varphi(S)^{\cdots\rho\cdots}_{\cdots\sigma\cdots}$$

Similar equality holds for pull-back.

Theorem. *Pull-back and push-forward is commutative with contraction, i.e. $\mathrm{d}\varphi \circ C^i_j = C^i_j \circ \mathrm{d}\varphi$. Similar equality holds for pull-back.*

Group of Diffeomorphism

Definition. ***One-parameter local group of diffeomorphism*** *of manifold M is a smooth map $\varphi : I \times U \to M$ where I is an open interval with $0 \in I$ and U is an open subset of M satisfying*

(i) $(\forall t \in I) :: \varphi_t : U \to \mathrm{img}(\varphi)[U]$ *is a diffeomorphism.*
(ii) $(\forall t, s, t+s \in I) :: \varphi_t \circ \varphi_s = \varphi_{t+s}$

where $\varphi_t(p) := \varphi(t, p)$.

Definition. ***One-parameter group of diffeomorphism*** *is a one-parameter local group of diffeomorphism satisfying $I = \mathbb{R}$ and $U = M$.*

Definition. ***Orbit*** *of the one-parameter group of diffeomorphism φ_p is defined as $\varphi_p(t) = \varphi(p, t)$ and $\varphi_p(0) = p$.*

Theorem. *One-parameter group of diffeomorphism φ can induce a smooth vector field v with $v|_p$ being the tangent vector of φ_p at $t = 0$.*

Theorem. *Smooth vector field v can induce a one-parameter local group of diffeomorphism φ with $\varphi_t(p)$ and p at the same integral curve of v and differ t in parameter.*

Definition. *Vector field is* ***complete*** *if the domain of all its integral curves is $\mathbb{R}$.*

Theorem. *Smooth complete vector field can induce a one-parameter group of diffeomorphism.*

Lie Derivative

Definition. ***Lie derivative*** *of tensor field $T^{\cdots\mu\cdots}_{\cdots\nu\cdots}$ along vector field v is defined as*

$$\mathcal{L}_v T^{\cdots\mu\cdots}_{\cdots\nu\cdots} := \lim_{t\to 0} \frac{1}{t}[\mathrm{d}\varphi_t(T)^{\cdots\mu\cdots}_{\cdots\nu\cdots} - T^{\cdots\mu\cdots}_{\cdots\nu\cdots}]$$

where φ is the one-parameter local group of diffeomorphism induced by vector field v.

Theorem. $\mathcal{L}_v f = v^\mu \partial_\mu f$

Definition. ***Isometry*** *of generalized Riemann space $(M, g_{\mu\nu})$ is a diffeomorphism $\varphi : M \to M$ satisfying*

$$\mathrm{q}\varphi(g)_{\mu\nu} = g_{\mu\nu}$$

Definition. ***Killing field*** *ξ is a vector field satisfying*

$$\mathcal{L}_\xi g_{\mu\nu} = 0$$

Theorem. *Killing field satisfies the following Killing equations*

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$$

Immersion and Embedding

Definition. ***Immersion*** *$\varphi : S \to M$ is a smooth injection satisfying*

(i) $\dim S \leqslant \dim M$
(ii) $(\forall p \in S, v \in \mathrm{T}_p S) :: \mathrm{d}\varphi(v) = 0 \Rightarrow v = 0$

Note. Map $\varphi : S \to \mathrm{img}(\varphi)[S]$ can be naturally defined as a diffeomorphism, where φ is an immersion.

Definition. ***Immersed submanifold*** *is the image of the immersion.*

Definition. ***Topological embedding*** *is an immersion φ with map $\varphi : S \to \mathrm{img}(\varphi)[S]$ being homeomorphic.*

Definition. ***Embedded submanifold*** *(or* ***regular submanifold****) is an immersed manifold with immersion being topological embedding.*

Definition. ***Hypersurface*** *S is an immersed submanifold satisfying $\dim S = \dim M - 1$*

Definition. *Vector v at q is* ***tangent*** *to hypersurface $\varphi[S]$ if it is the tangent vector of a curve of $\varphi[S]$.*

Definition. ***Normal covector*** *n_μ at q is a dual vector in $\mathrm{T}^*_q\varphi[S]$ satisfying $(\forall w^\mu \in \mathrm{T}_q\varphi[S]) :: w^\mu n_\mu = 0$*

Theorem. *Any covector w_μ and u_μ satisfies $w_\mu = h u_\mu$ where $h \in \mathbb{R}$.*

Definition. ***Normal vector*** *n^μ of hypersurface $\varphi[S]$ in generalized Riemann space $(M, g_{\mu\nu})$ is defined as $n^\mu = g^{\mu\nu} n_\mu$.*

Definition. *Hypersurface is*

(i) ***Time-like*** *if $n_\mu n^\mu > 0$*
(ii) ***Space-like*** *if $n_\mu n^\mu < 0$*
(iii) ***Light-like*** *if $n_\mu n^\mu = 0$*

Note. If normal vector satisfies $n_\mu n^\mu \neq 0$, there is a normalized normal vector such that $|n_\mu n^\mu \neq 0| = 1$. In the following context, default normal vector is normalized if possible.

Definition. ***Induced metric*** *$h_{\mu\nu}$ at q of hypersurface $\varphi[S]$ of $(M, g_{\mu\nu})$ is a metric on $\varphi[S]$ satisfying*

$$(\forall w^\mu, u^\nu \in \mathrm{T}_q\varphi[S]) :: h_{\mu\nu} w^\mu u^\nu = g_{\mu\nu} w^\mu u^\mu$$

Definition. *For time-like and space-like hypersurface, induced metric is extended and overloaded as*

$$h_{\mu\nu} = g_{\mu\nu} + (-1)^s n_\mu n_\nu$$

where $s = n^\mu n_\mu$ and n^μ is the normal vector.

6.6 Measure Structure

This section concerns the differential form and integration on manifolds. Default l-form is overloaded as differential l-form field. Default measure is adapted measure.

Differential Form

Definition. ***l-form*** *of vector space V is a totally antisymmetric tensor of type $(0, l)$ on V.*

Note. All l-form on V is denoted by $\Lambda(l)$.

Definition. ***Wedge product*** *of l-form ω and m-form η is a $l + m$-form defined as*

$$(\omega \wedge \eta)_{\dots\mu\dots\nu\dots} := \frac{(l+m)!}{l!m!}\omega_{[\dots\mu\dots}\eta_{\dots\nu\dots]}$$

Theorem. *If* $\dim V =$*, then* $\Lambda(l) = \{0\}$ *for $l > n$.*

Note. Thus, only $l \leqslant n$ case is considered.

Theorem. *If* $\dim V = n$*, then* $\dim\Lambda(l) = n!/(l!(n - l!))$.

Proof. Set $\{e^a\}$ a dual basis of $\mathcal{T}_V(0, l)$. Notice that

$$\{\bigwedge_{i\in\mathbb{Z}_l} e_i^{a_i}\} \text{ with } \bigwedge_{i\in\mathbb{Z}_m}\omega^i = \bigwedge_{i\in\mathbb{Z}_{m-1}}\omega^i \wedge \omega^m, \bigwedge_{i\in\mathbb{Z}_1}\omega^i = \omega^1$$

where $\{a_i\}$ is a permutation of $\mathbb{Z}_n$ and $a_i < a_j$ if $i < j$ is a basis of $\Lambda(l)$. There are in total $n!/(l!(n - l)!)$ elements in the basis, and hence the theorem is proved. □

Definition. ***l-form field*** *on manifold M is a totally symmetric tensor field of type $(0, l)$ on M.*

Definition. ***Differential l-form field*** *is a smooth l-form field.*

Exterior Derivative

Definition. ***Exterior derivative*** *is a map* $\mathrm{d} : \Lambda(l) \to \Lambda(l+1)$ *defined as*

$$\mathrm{d}\omega_{\mu\dots\nu\dots} = (l + 1)\,\partial_{[\mu}\omega_{\dots\nu\dots]}$$

where ∂_μ is any connection.

Note. For $f \in \Lambda(0)$, there is $\mathrm{d}f_\mu = \partial_\mu f$.

Theorem. $\mathrm{d} \circ \mathrm{d} = 0$

Definition. *l-form ω is* ***closed*** *if* $\mathrm{d}\omega = 0$.

Definition. *l-form ω is* ***exact*** *if* $(\exists\eta\in\Lambda(l-1) :: \omega = \mathrm{d}\eta$.

Theorem. *If l-form ω is exact, then it is closed.*

Note. The inverse predicate does not always hold.

Measure and Integration

Definition. *n-dimensional manifold is* ***orientable*** *if there is a continuous non-zero n-form.*

Definition. ***Equivalence relation of orientation*** $\sim$ *on manifold M is defined as*

$$\varepsilon \sim \varepsilon' \Leftrightarrow (\exists h \in \mathcal{F}(M, \mathbb{R}_+)) :: \varepsilon = h\varepsilon'$$

where ε and ε' is a continuous non-zero n-form.

Definition. ***Orientation*** *of n-dimensional orientable manifold is the element of $\{\varepsilon\}/\!\sim$ where ε is a continuous non-zero n-form.*

Note. Orientation is a mathematical structure.

Note. Manifold M is **oriented** if there is a orientation defined.

Definition. *Basis $\{e_a^\mu\}$ on open set O in oriented manifold M is* ***right-handed*** *if*

$$(\exists h > 0) :: \varepsilon = h \bigwedge_{i\in\mathbb{Z}_n} e^i$$

where ε is the orientation and $\{e_\mu^i\}$ is the dual basis of basis $\{e_a^\mu\}$.

Definition. *Basis $\{e_a^\mu\}$ on open set O in oriented manifold M is* ***left-handed*** *if it is not right-handed.*

Definition. *Coordinate system (O, ψ) is* ***right-handed*** *(*resp *left-handed) if its coordinate basis is right-handed (*resp *left-handed).*

Definition. ***Integral*** *of continuous n-form ω on open set G is defined as*

$$\int_G \omega := \int_{\psi[G]} \omega(x)\,\mathrm{d}\mu$$

where (O, ψ) is a coordinate system with $G \subset O$, $\omega(x)$ is the coefficient of ω under coordinate dual basis and μ is the usual measure.

Definition. ***Measure*** *of n-dimensional orientable manifold M is a continuous non-zero n-form.*

Definition. ***Adapted measure*** *of n-dimensional orientable generalized Riemann space $(M, g_{\mu\nu})$ is a continuous non-zero n-form ε satisfying*

$$g^{\mu\nu}\varepsilon_{\dots\mu\dots}\varepsilon_{\dots\nu\dots} = (-1)^s n!$$

where s is half of the signature of the metric.

Definition. ***Integral*** *of continuous function f on G in $(M, g_{\mu\nu})$ is defined as*

$$\int_G f\,\mathrm{d}\mu := \int_G f\varepsilon$$

7 Lie Group & Lie Algebra

This chapter elucidates Lie group and Lie algebra theory.

7.1 Lie Group

Definition. ***n-dimensional Lie group*** *is a set that is both a manifold and a group satisfying*

(i) Group multiplication $\cdot : g \cdot h \mapsto gh$ is smooth.
(ii) Inversion map $-1 : g \mapsto g^{-1}$ is smooth.

Definition. ***Subgroup*** *H of group G is a subset of G closed under group multiplication.*

Definition. ***Homomorphism*** *μ of group G and G' is a map $\mu : G \to G'$ satisfying*

$$(\forall g, h \in G) :: \mu(gh) = \mu(g)\mu(h)$$

Note. The next two definitions are overloaded.

Definition. ***Isomorphism*** *is a bijective homomorphism.*

Definition. ***Automorphism*** *is a homomorphism $\mu : G \to G$.*

Definition. ***Direct product group*** *$G \times G'$ is a group with multiplication defined as*

$$(g, h) \cdot (g', h') := (gg', hh')$$

Definition. ***Lie group homomorphism*** *is a smooth homomorphism between Lie groups.*

Definition. ***Lie group isomorphism*** *is a diffeomorphic homomorphism.*

Definition. ***Lie subgroup*** *H of Lie group G is a subset of G being both the subgroup and immersed submanifold of G.*

Definition. ***Left translation generated by*** *$g \in G$ is a map L_g defined as*

$$L_g : h \mapsto gh$$

Definition. *Vector field $\bar{X}$ on Lie group is **left-invariant** if*

$$\mathrm{d}L_g(\bar{X}) = \bar{X}$$

Theorem. *The set of all left-invariant vector field $\mathcal{L}(G)$ of Lie group G becomes a vector space under addition and scalar multiplication defined pointwise.*

Theorem. *Vector space $\mathrm{T}_e G$ is isomorphic to $\mathcal{L}(G)$.*

7.2 Lie Algebra

Definition. ***Lie bracket*** *on vector space V is a map $[,] : V \times V \to V$ satisfying*

(i) Bilinearity $(\forall u, v, w \in V, \alpha, \beta \in \mathbb{R})$

$$[\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w]$$
$$[u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w]$$

(ii) Anticommutativity

$$(\forall u, v \in V) :: [u, v] = -[v, u]$$

(iii) Jacobi identity

$$(\forall u, v, w \in V) :: [u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

Definition. ***Lie algebra*** *is a vector space $\mathscr{G}$ with Lie bracket defined.*

Theorem. *$\mathcal{L}(G)$ becomes a Lie algebra under*

$$(\forall \bar{X}, \bar{Y} \in \mathcal{L}(G)) :: [\bar{X}, \bar{Y}] := \bar{X} \circ \bar{Y} - \bar{Y} \circ \bar{X}$$

Definition. ***Lie algebra homomorphism*** *$\beta : \mathscr{V} \to \mathscr{W}$ is a linear map satisfying*

$$(\forall u, v \in \mathscr{V}) :: \beta([u, v]) = [\beta(u), \beta(v)]$$

Definition. ***Lie subalgebra*** *is a subset of Lie algebra closed under the Lie bracket.*

Definition. ***Left-invariant vector field*** *$\bar{X}$ of Lie group G corresponding to vector $X \in \mathrm{T}_e G$ is defined as*

$$\bar{X}|_g := \mathrm{d}L_g X$$

Definition. ***Lie algebra $\mathscr{G}$ of Lie group*** *G is the vector space $\mathrm{T}_e G$ with Lie bracket defined as*

$$[X, Y] := [\bar{X}, \bar{Y}]|_e$$

Definition. ***Generator*** *of Lie group G is a basis of corresponding Lie algebra.*

Theorem. *If $\rho : G \to \hat{G}$ is a Lie group homomorphism, then $\mathrm{d}\rho|_e : \mathscr{G} \to \hat{\mathscr{G}}$ is a Lie algebra homomorphism.*

Theorem. *If H is the Lie subgroup of G, then $\mathscr{H}$ is the Lie subalgebra of $\mathscr{G}$.*

7.3 Exponential Map

Definition. ***Homomorphic line*** *is a smooth map* $\gamma:\mathbb{R}\to G$ *satisfying*

$$\gamma(s+t)=\gamma(s)\gamma(t)$$

Theorem. *There is a natural bijection between homomorphic line and the integration curve of left-invariant vector field.*

Definition. ***Exponential map*** $\exp : \mathrm{T}_eG \to G$ ***of Lie group*** *G is defined as*

$$\exp(X) := \gamma(1)$$

where γ *is the corresponding homomorphic line of* $\bar{X}$.

Theorem. $(\forall s\in\mathbb{R}, X\in\mathrm{T}_eG) :: \exp(sX)=\gamma(s)$

Note. Thus, $\exp(sX)$ is utilized to denote a homomorphic line.

7.4 Representation Theory

Definition. ***Lie group of transformation*** *of G on manifold M is a smooth map* $\sigma : G\times M\to M$ *satisfying*

(i) $(\forall g\in G) :: \sigma_g : M\to M$ *is a diffeomorphism.*
(ii) $(\forall g,h\in G) :: \sigma_{gh}=\sigma_g\circ\sigma_h$

where $\sigma_g(x) := \sigma(g,x)$.

Definition. ***Realization*** *of Lie group G on manifold M is a homomorphism*

$$\rho : g\mapsto\sigma_g$$

Definition. ***Realization space*** *is the manifold M in the above definition.*

Definition. *Realization is* ***faithful*** *if it is a isomorphism.*

Definition. ***Representation*** ρ *of Lie group G on manifold M is a realization of Lie group G on manifold M with M being a vector space.*

Note. Representation is sometimes overloaded as its image.

Definition. ***Representation space*** *is the realization space of representation.*

Definition. *Representation is* ***faithful*** *if it is a isomorphism.*

7.5 Matrix Groups

This section illustrates several specific matrix groups.

General Linear Group

Definition. ***General linear group*** $\mathrm{GL}(m,F)$ ***of degree*** m *is the set of all invertible endomorphism of m-dimensional vector space V over field* $F\in\{\mathbb{R},\mathbb{C}\}$ *with map composition as group multiplication.*

Definition. ***Special linear group*** $\mathrm{SL}(n,F)$ *is a subgroup of* $\mathrm{GL}(n,F)$ *satisfying* $(\forall X\in\mathrm{SL}(n,F)) :: \det(X)=1$ *where the determinant of X is the determinant of matrix of X under any basis.*

Note. The above definition is valid since the determinant of linear map is unique.

Theorem. *Lie algebra* $\mathscr{GL}(m,F)$ *of* $\mathrm{GL}(m,F)$ *is the set of all square matrix of order n of field F.*

Theorem. $\dim\mathrm{GL}(n,F)=\dim\mathscr{GL}(n,F)=n^2$

Definition. ***Exponential function*** Exp *of square matrix X of order m is defined as*

$$\mathrm{Exp}(X) := I+\sum_{i\in\mathbb{Z}_+}\frac{1}{i!}X^i$$

where I is the identity matrix and $A^i=AA^{i-1}, A^1=A$.

Theorem. $(\forall X\in\mathscr{GL}(m,F)) :: \exp(X)=\mathrm{Exp}(X)$

Definition. ***Matrix group*** *refers to general linear group and all its subgroups.*

Theorem. *The Lie bracket of matrix group is*

$$(\forall X,Y\in\mathscr{GL}(m,F)) :: [X,Y]=XY-YX$$

Orthogonal Group

Definition. *Endomorphism* Z^μ_ν *of vector space V with metric* $g_{\mu\nu}$ *is* ***metric-preserved*** *if*

$$g_{\mu\nu}Z^\mu_\rho Z^\nu_\sigma=g_{\rho\sigma}$$

Definition. ***Orthogonal Group*** $\mathrm{O}(m)$ *is a subgroup of* $\mathrm{GL}(m,F)$ *on vector space V with positive definite metric* $g_{\mu\nu}$ *and all elements being metric-preserved.*

Definition. ***Special orthogonal group*** $\mathrm{SO}(m)$ *is a subgroup of* $\mathrm{O}(m)$ *satisfying* $(\forall Z\in\mathrm{SO}(m)) :: \det(Z)=1$.

Theorem. *Lie algebra* $\mathscr{O}(m)$ *of* $\mathrm{O}(m)$ *is the set of square matrix X of order n satisfying* $\mathrm{tsp}(X)=-X$.

Theorem. $\dim\mathrm{O}(m)=\dim\mathscr{O}(m)=m(m-1)/2$

Lorentz Group

Definition. ***Lorentz Group*** *$L(1, n-1)$ is a subgroup of* $\mathrm{GL}(n, F)$ *on vector space V with Lorentzian metric $g_{\mu\nu}$ and all elements being metric-preserved.*

Definition. ***Proper Lorentz group*** *$L_+^\uparrow$ is a subgroup of $L(1, n-1)$ satisfying* $(\forall \Lambda \in L_+^\uparrow) :: \det(\Lambda) = 1$.

Theorem. *The proper Lorentz group is connected.*

Theorem. *Lie algebra $\mathscr{L}(1, n-1)$ of Lorentz group is the set of square matrix Λ of order n satisfying* $\mathrm{tsp}(\Lambda) = -\eta\Lambda\eta$ *where* $\eta_a^b := -\delta_a^0\delta_0^b + \delta_i^j\delta_a^i\delta_j^b$.

Note. Matrix η_a^b defined above is phrased Minkowski matrix.

Theorem. $\dim L(1, n-1) = \dim L_+^\uparrow + 1 = n$

Unitary Group

Definition. ***Unitary operator*** *on inner product space V is an endomorphism of V satisfying*

$$(\forall f, g \in V) :: (Uf, Ug) = (f, g)$$

Definition. ***Adjoint operator*** *of endomorphism U on inner product space V is an endomorphism $U^\dagger$ satisfying*

$$(\forall f, g \in V) :: (U^\dagger f, g) = (f, Ug)$$

Theorem. *Unitary operator satisfies* $U^\dagger U = \mathrm{id}_V$.

Definition. ***Unitary matrix*** *U is a complex square matrix satisfying* $\mathrm{tsp}(\bar{U})U = U^\dagger U = I$.

Definition. ***Unitary group*** $\mathrm{U}(m)$ *is the set of all unitary operator on m-dimensional inner product space with map composition as group multiplication.*

Theorem. *Unitary group is compact and connected.*

Definition. *Complex square matrix U is* ***hermitian*** *(*resp *anti-hermitian) if $U^\dagger = U$ (*resp *$U^\dagger = -U$).*

Theorem. *For matrix U,* $\det(\mathrm{Exp}(U)) = \exp(\mathrm{tr}(U))$.

Theorem. *Lie algebra $\mathscr{U}(m)$ of unitary group* $\mathrm{U}(m)$ *is the set of complex anti-hermitian matrix of order m.*

Definition. ***Special unitary group*** $\mathrm{SU}(m)$ *is a subgroup of* $\mathrm{U}(m)$ *satisfying* $(\forall U \in \mathrm{SU}(m)) :: \mathrm{tr}U = 0$ *where the trace of U is the trace of matrix of U under any basis.*

Theorem. $\dim\mathrm{U}(m) = \dim\mathscr{U}(m) = m^2$

8 Fibre Bundle

This chapter elaborates the fibre bundle theory.

8.1 Principal Bundle

Definition. ***Left action*** *of Lie group G on manifold K is a smooth map $L : G \times K \to K$ satisfying*

(i) $L_g : K \to K$ is a diffeomorphism.
(ii) $L_{gh} = L_g \circ L_h$

where $(\forall p \in K) :: L_g(p) = L(g, p)$.

Definition. ***Right action*** *of Lie group G on manifold K is a smooth map $L : K \times G \to K$ satisfying*

(i) $R_g : K \to K$ is a diffeomorphism.
(ii) $R_{gh} = L_h \circ L_g$

where $(\forall p \in K) :: R_g(p) = R_p(g) = R(p, g)$.

Note. Now, denote the left action $L_g(p)$ as gp and right action $R_g(p)$ as pg.

Definition. *Right action is* ***free*** *if $g \neq e \Rightarrow pg \neq p$.*

Definition. ***Principle fibre bundle*** *$P(M, G)$ (or* ***principle bundle****) consists of a* ***bundle manifold*** *P,* ***base manifold*** *M and structure group G with*

(i) Free right action $R : P \times G \to P$.
(ii) Projection map $\varrho : P \to M$ satisfying

$$(\forall p \in P) :: \varrho^{-1}[\varrho(p)] = \{\, pg \mid g \in G \,\}$$

(iii) Local trivialization $T_U : \varrho^{-1}[U] \to U \times G$ for every $x \in M$ and $U \in \mathcal{N}(x)$ as

$$(\forall p \in P) :: T_U(p) = (\varrho(p), S_U(p))$$

with map $S_U : \varrho^{-1}[U] \to G$ satisfying

$$(\forall g \in G) :: S_U(pg) = S_U(p)g$$

Definition. ***Fibre*** *of P over $x \in M$ is the set* $\varrho^{-1}[\{x\}]$.

Definition. ***Characteristic point*** *$\tilde{p}$ of $\varrho^{-1}[x]$ is a point satisfying $S_U(\tilde{p}) = e$.*

Note. Symbol $\varrho^{-1}[x]$ refers to $\varrho^{-1}[\{x\}]$.

Definition. ***Transition function*** *g_{UV} from local trivialization T_U to T_V is defined as*

$$g_{UV}(x) = S_U(p)S_V(p)^{-1}$$

where $x = \varrho(p)$.

Definition. ***Local cross section*** *$\sigma(x)$ on open set U in principal bundle $P(M,G)$ is a smooth map $\sigma : U \to P$ satisfying*

$$(\forall x \in U) :: \varrho(\sigma(x)) = x$$

Theorem. *There is an injection between local trivialization and local cross section.*

Note. The set of all basis of vector space V is denoted by $\operatorname{basis} V$.

Definition. ***Frame bundle*** *$\mathrm{F}M$ of manifold M is constructed as*

(i) Bundle set

$$P = \{(x, \{e^\mu_a\} \mid x \in M, e^\mu_a \in \operatorname{basis} \mathrm{T}_x M\}$$

(ii) Bundle manifold (differential structure of bundle set) $(\hat{O}, \hat{\psi})$ as

$$\hat{O} := \{(x, e^\mu_a) \mid x \in O,\ e^\mu_a \in \operatorname{basis} \mathrm{T}_x M\}$$
$$\hat{\psi}(x, e^\mu_a) := (\psi(x), e^b{}_a)$$

where (O, ψ) is a chart of M, (x, e^μ_a) refers to $(x, \{e^\mu_a\})$, and $e^b{}_a$ is the coordinate component of the basis.

(iii) Structure group $G = \mathrm{GL}(\dim M, \mathbb{R})$.

(iv) Right action

$$R_g(x, e^\mu_a) := (x, e^\mu_a g^a{}_b)$$

(v) Projection map

$$\varrho(x, e^\mu_a) := x$$

(vi) Local Trivialization

$$T_U(x, e^\mu_a) = (x, h)$$

and $h = S_U(x, e^\mu_a)$ satisfies $X^\mu_a h^a{}_b = e^\mu_b$ where $\{X^\mu_a\}$ is the coordinate basis of chart (O, ψ) with $U \subset O$.

Note. Construct can refer to define.

Definition. ***Orthogonal frame bundle*** *of spacetime $(M, g_{\mu\nu})$ is a frame bundle with basis being orthonormal and structure group being Lorentz group.*

Note. Default frame bundle of spacetime is orthogonal frame bundle.

Definition. ***Vertical subspace*** *V_p is a subspace of $\mathrm{T}_p P$ defined as*

$$\mathrm{V}_p := \{X \in \mathrm{T}_p P \mid \mathrm{d}\varrho(X) = 0\}$$

Definition. ***Vertical vector*** *is the element of vertical space.*

Definition. ***Fundamental vector field*** *$\tilde{X}$ induced by $X \in \mathscr{G}$ of principal bundle $P(M,G)$ is defined as*

$$(\forall p \in P) :: \tilde{X}|_p := \mathrm{d}R_p(X)$$

8.2 Associated Bundle

Definition. ***Product manifold*** *$M \times N$ of manifold M and N is a set $M \times N$ with product topology and differential structure (O, ψ) defined as*

$$(\forall (m,n) \in O) :: \psi(m,n) := (\varphi(m), \phi(n))$$

where (U, φ) is a chart of M and (V, ϕ) is a chart of N.

Definition. ***Associated bundle*** *$Q = P \times F/\sim$ of principal bundle $P(M,G)$ consists of a **typical fibre** manifold F and left action of G on F with*

(i) Bundle set $Q = P \times F/\sim$ where equivalence relation is defined as

$$(\forall (p,f) \in P \times F) :: (p,f) \equiv p \cdot f \sim pg \cdot g^{-1} f$$

(ii) Projection map

$$(\forall q \equiv p \cdot f \in Q) :: \tilde{\varrho}(q) := \varrho(p)$$

(iii) Local trivialization

$$(\forall q \in \tilde{\varrho}^{-1}[U]) :: \tilde{T}_U(q) := (\tilde{\varrho}(q), \tilde{f})$$

where $q \equiv \tilde{p} \cdot \tilde{f}$ with $\tilde{p}$ being the characteristic point of local trivialization of principal bundle.

(iv) Bundle manifold (differential structure of bundle set) $(\hat{O}, \hat{\psi})$ as

$$\hat{\psi} := \psi \circ \tilde{T}_U$$

where (O, ψ) is a chart of $U \times F$.

Definition. ***Local cross section*** *$\tilde{\sigma}(x)$ on open set U in associated bundle Q is a smooth map $\tilde{\sigma} : U \to Q$ satisfying*

$$(\forall x \in U) :: \tilde{\varrho}(\sigma(x)) = x$$

Note. Default cross section is local cross section.

Definition. ***Fibre*** *of Q over $x \in M$ is the set $\tilde{\varrho}^{-1}[\{x\}]$.*

8.3 Physical Field

Definition. ***Gamma matrices*** *are a set of matrices* $\{\gamma_c\}$ *satisfying*

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab} I$$

where η_{ab} *is the Minkowski matrix and* I *is the identity matrix.*

Definition. ***Hermiticity condition*** *is defined as*

$$(\gamma^a)^\dagger = \gamma^0\gamma^a\gamma^0$$

Note. Default gamma matrices are chosen to satisfy hermiticity condition.

Note. Define γ_μ as $\gamma_\mu := \gamma_a \varepsilon^a_\mu$ where $\gamma_a \equiv \eta_{ab}\gamma^b$ and $\{\varepsilon^a_\mu\}$ is the dual basis field on some manifold.

Note. Define $\sigma_{\mu\nu}$ as

$$\sigma_{\mu\nu} := \frac{1}{2}\mathrm{i}[\gamma_\mu, \gamma_\nu]$$

where $[\gamma_\mu, \gamma_\nu] := \varepsilon^a_\mu[\gamma_a, \gamma_b]\varepsilon^b_\nu$.

Note. Define $S_{\mu\nu}$ as

$$\varepsilon^\mu_c \varepsilon^\nu_d (S_{\mu\nu})^a{}_b := -\mathrm{i}\delta^a_c \eta_{bd} + \mathrm{i}\delta^a_d \eta_{bc}$$

Definition. ***Spin-zero field*** φ *is a cross section of associated bundle* $\mathrm{F}M \times \mathbb{C}/\sim$ *of spacetime frame bundle with left action defined as*

$$(\forall f \in \mathbb{C}, g \in L) :: g^{-1} f := f$$

Definition. ***Spin-half field*** ψ *is a cross section of associated bundle* $\mathrm{F}M \times V/\sim$ *of spacetime frame bundle with inner product space* V *satisfying* $\dim V = \dim M$ *and left action defined as*

$$(\forall f_a \in V, g \equiv \exp\{-\frac{1}{2}\mathrm{i}\tau^{\mu\nu}S_{\mu\nu}\}) :: g^{-1} f_a := \exp\{\frac{1}{4}\mathrm{i}\tau^{\mu\nu}\sigma_{\mu\nu}\}^b_a f_b$$

where $\tau^{\mu\nu} \equiv \tau^{ab}\varepsilon^\mu_a \varepsilon^\nu_b$ *with* τ^{ab} *being the constant coordinates of* g*, and the inner product is defined as*

$$(f, g) := \bar{f}^a (\gamma^0)_a{}^b g_b$$

where $\bar{f}^a \equiv \eta^{ab}\bar{f}_b$ *and* $\bar{f}_b$ *refers to complex conjugate.*

Definition. ***Spin-one field*** ω *is a cross section of associated bundle* $\mathrm{F}M \times \Lambda/\sim$ *of spacetime frame bundle with inner product space* Λ *satisfying* $\dim \Lambda = \dim M$ *and left action defined as*

$$(\forall f_a \in \Lambda, g \equiv \exp\{-\frac{1}{2}\mathrm{i}\tau^{\mu\nu}S_{\mu\nu}\}) :: g^{-1} f_a := \exp\{\frac{1}{2}\mathrm{i}\tau^{\mu\nu}S_{\mu\nu}\}^b{}_a f_b$$

And the inner product is defined as $(f, g) := \eta_{ab} f^a g^b$.

Definition. ***Physical field*** *refers to spin-zero, spin-half and spin-one field.*

Note. The binary operation φ (addition and scalar multiplication) of elements $q \equiv p \cdot f$ and $r \equiv p \cdot g$ in fibre of associated bundle is defined as

$$q\varphi r := p \cdot (f\varphi g)$$

Note. The binary operation (addition and scalar multiplication) of spin-zero, spin-half and spin-one field is defined pointwise.

Note. The set of all spin-zero, spin-half and spin-one field become vector spaces under the above binary operation respectively.

Definition. ***Field representation*** *is the left action on the physical field.*

Definition. ***Conjugate spin-half field*** $\bar{\psi}$ *of spin-half field* ψ *is the natural dual of* ψ *induced by the inner product pointwise.*

Theorem. *There is a natural isomorphism between the space of all real spin-one field and the space of* 1*-form.*

8.4 Connection on Bundle

Definition. $\mathscr{V}$***-valued tensor field*** $\mathcal{T}$ *with* $\mathscr{V}$ *being a vector space is defined as* $\mathcal{T} := Tv \equiv T \otimes v$ *where* T *is a tensor field and* $v \in V$ *is a vector.*

Definition. ***Adjoint isomorphism*** I_g *of group* G *induced by* $g \in G$ *is defined as*

$$(\forall h \in G) :: I_g(h) := g^{-1}hg$$

Definition. ***Connection*** ϖ *of principal bundle* $P(M, G)$ *is a* $\mathscr{G}$*-valued spin-one field satisfying*

(i) $(\forall \hbar \in \mathscr{G}) :: \varpi_\mu|_p \tilde{\hbar}^\mu|_p = \hbar$

(ii) $(\forall X \in \mathrm{T}_p P, g \in G) :: \varpi_\mu|_{pg} \mathrm{d}R_g(X)^\mu = \mathrm{d}I_g(\varpi_\mu|_p X^\mu)$

Note. Map f, g between vector spaces with domain being V and W respectively can be extended to $v \otimes w$ as

$$f(v \otimes w) := f(v) \otimes w, \quad g(v \otimes w) := v \otimes g(w)$$

where $v \in V$ and $w \in W$. Furthermore, map f and g can be defined pointwise for tensor fields.

Note. The above extension is automatically applied in the following context.

Definition. ***Connection*** *ω on U of local trivialization T_U is defined as $\omega := \mathrm{q}\sigma(\varpi)$ where σ is the corresponding local cross section.*

Theorem. *The connections of local trivialization T_U and $T_{U'}$ satisfy*

$$(\forall x \in U \cap U', X \in \mathrm{T}_x M) :: \omega'_\mu X^\mu = \mathrm{d}I_g(\omega_\mu X^\mu) + \mathrm{d}L_g^{-1} \circ \mathrm{d}g(X)$$

where g is the transition function from T_U to $T_{U'}$. And for structure group G being matrix group, the relation simplifies to

$$\omega' = g^{-1}\omega g + g^{-1}\mathrm{d}g$$

where the product of matrix group element g and Lie algebra $\mathfrak{g}$ is defined as composite of corresponding linear map. And exterior derivative $\mathrm{d}g$ is defined as

$$\mathrm{d}g := \mathrm{d}g^r e_r$$

where $\{e_r\}$ is a basis of Lie algebra $\mathscr{G}$.

Definition. ***Covariant derivative*** *$\mathcal{D}_\mu$ of physical field $\tilde{\sigma}$ is defined as*

$$\mathcal{D}_\mu \tilde{\sigma} := \sigma \cdot [\partial_\mu f + \mathrm{d}\rho(\omega)_\mu f]$$

where σ is a cross section of frame bundle and ρ is the representation of structure group on typical fibre.

8.5 Curvature of Connection

Definition. ***Graded wedge product*** *of $\mathscr{G}$-valued n-form $\eta\mathfrak{g}$ and $\lambda\mathfrak{h}$ is defined as*

$$[\eta\mathfrak{g} \wedge \lambda\mathfrak{h}] := (\eta \wedge \lambda)\,[\mathfrak{g}, \mathfrak{h}]$$

Definition. ***Exterior covariant derivative*** *$\mathcal{D}$ of $\mathscr{G}$-valued n-form η is defined as*

$$\mathcal{D}\eta := \mathrm{d}\eta + \frac{1}{2}[\varpi \wedge \eta]$$

Definition. ***Curvature*** *$\widetilde{\Omega}$ of connection ϖ is defined as*

$$\widetilde{\Omega} := \mathcal{D}\varpi \equiv \mathrm{d}\varpi + \frac{1}{2}[\varpi \wedge \varpi]$$

Definition. ***Curvature*** *Ω on U of local trivialization T_U is defined as $\Omega := \mathrm{q}\sigma(\widetilde{\Omega})$ where σ is the corresponding local cross section.*

Theorem. *For matrix structure group, there is*

$$\widetilde{\Omega} = \mathrm{d}\varpi + \varpi \wedge \varpi$$

Note. The curvature of connection of frame bundle coincides with the Riemann curvature.

9 Gauge Field Theory

This chapter constructs the foundations of Physics and the classical gauge field theory.

9.1 Background Setup

The background elements of gauge field theory are constructed as follows

Topological space M with

Separation	Hausdorff
Countability	Second-countable
Connectedness	Connected

Base manifold M with

Dimension	n or 4
Differentiability	Smooth
Orientability	Orientable

Structure Group G with

Element	Matrix group
Differentiability	Lie group
Representation ρ	Faithful
and space V	Inner product space

Principal bundle $P = M \times G$ with

Right action	$(x, g)h := (x, gh)$
Projection	Natural projection
Trivialization	Identity map

Typical fibre F with

Algebra	Banach space
Differentiability	Smooth

Associated bundle $Q = P \times F / \sim$ with

Left action	Representation

Structure $[P(M,G), Q]$ forms the background of gauge field theory.

9.2 Field Construction

Definition. ***Matter field*** *$\hat{\sigma}$ is the cross section of Q with fibre over x constructed as $\tilde{\sigma}(x) \cdot v \equiv \tilde{\sigma}(x) \otimes v$ where $v \in V$ and $\tilde{\sigma}$ is a spin-zero or spin-half field.*

Definition. ***Gauge field*** *ω is a connection of $P(M,G)$.*

9.3 Symmetric Principle

Definition. ***Action*** *$\mathcal{I}[\psi, \partial_\mu]$ is a real continuous functional of physical fields.*

Note. Physical field is now overloaded as matter field and gauge field.

Definition. *Functional $\mathcal{H}[\psi, \partial_\mu\psi]$ is* ***local*** *if*

$$\mathcal{H}|_p = \mathcal{H}[\psi|_p, \partial_\mu\psi|_p]$$

Definition. ***Lagrangian*** *$\mathcal{L}[\psi, \partial_\mu\psi]$ is a local continuous functional of physical fields.*

Locality Condition. *The action has the formulation*

$$\mathcal{I} = \int \varepsilon\mathcal{L}$$

where the integration is on arbitrary open set of base manifold.

Definition. *Inner product of matter field $\hat{\sigma} \equiv \tilde{\sigma} \cdot v$ is defined as*

$$\langle\hat{\sigma}, \hat{\sigma}\rangle := \int \varepsilon(x)\,(\tilde{\sigma}(x), \tilde{\sigma}(x)) \cdot (v, v)$$

where the integration is on the same open set as the action.

Definition. *Inner product of gauge field $\omega \equiv \eta g$ is defined as*

$$\langle\omega, \omega\rangle := \int \varepsilon\,(\eta, \eta) \cdot \mathrm{tr}(g)$$

where the integration is on the same open set as the action.

Theorem. *The set of matter field or gauge field becomes a Banach space under the induced norm of inner product defined above.*

Definition. ***Interior variation operator*** *$\imath\delta$ is defined as*

$$\imath\delta\mathcal{H}[\psi] = \lim_{i\to\infty} \delta\mathcal{H}_i[\psi]$$

where $\mathcal{H}$ is a functional and the corresponding variation on ψ is

$$\imath\delta\psi = \lim_{i\to\infty} \varphi_i, \qquad \varphi_i\Big|_{\partial U} = 0$$

where U is the same open set as the action.

Symmetric Principle. *The* ***Symmetric Principle*** *is formulated as*

$$\imath\delta\mathcal{I} = 0$$

where $\mathcal{I}$ is the action and $\imath\delta$ is the internal variation.

Theorem. *The symmetric principle infers the following equation of motion*

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = 0$$

9.4 First Quantization

First Quantization is to construct the Lagrangian of spin-zero matter field φ as

$$\mathcal{L} = -\mathcal{D}^\mu\varphi^\dagger\mathcal{D}_\mu\varphi - m^2\varphi^\dagger\varphi$$

free spin-half matter field as

$$\mathcal{L} = -\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

free spin-one gauge field as

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}\Omega_{\mu\nu}\Omega^{\mu\nu}$$

interaction of spin-half and spin-one field as

$$\mathcal{L} = -\bar{\psi}\gamma^\mu\omega_\mu\psi$$

9.5 Conserved Charge

Definition. *Action $\mathcal{I}[\psi, \partial_\mu\psi]$ is* ***invariant*** *under transformation $\mathcal{L}_\xi$ if*

$$\mathcal{L}_\xi\mathcal{I} = \int \partial_\mu\ell^\mu\varepsilon$$

Noether's Theorem. *Every continuous symmetry in a theory $\mathcal{L}_\xi\psi$ corresponds to a conserved current*

$$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\mathcal{L}_\xi\psi - \ell^\mu$$

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